Front propagation in a transport model with a nonlocal nonlinear condition at the boundary

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Joint work with G. Faye and M. Zhang

Model and question

Unknown function: $u(t,x,i)$; $t>0$, $i>0$, $x\in\mathbb{R}$

$$\left\{\begin{array}{rll}\partial_t u+\partial_i u= & -\gamma(i)u & (t>0,x\in\mathbb{R},i>0)\\ u(t,x,0)= & f\left[\displaystyle\int_0^{+\infty}\beta(i)K*_x u(t,.,i)\ di\right].\end{array}\right.$$

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- $f(0)=0;\ f(u)>0,\ f''(u)<0$ for $u>0$.
- $K(x)\geq 0,\ C^\infty$, even, compact support, unit mass.
- $\beta(i)\geq 0,\ \gamma(i)>0$, both C^∞ w. compact support.

Sharp asymptotic behaviour $t\to+\infty$?

Motivations

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Outline

— The Fisher-KPP equation

— Derivation of the model

— What to expect, kown results

— Sharp asymptotic behaviour

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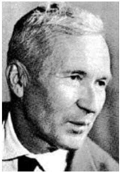

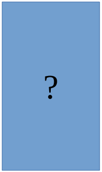

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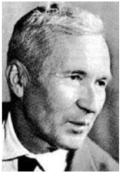

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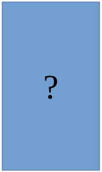

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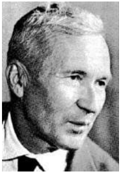

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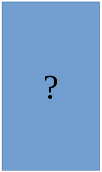

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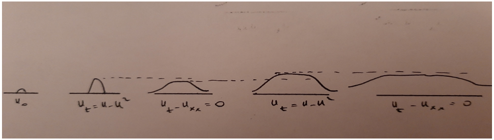

Invasion of whole line by the state $u = 1$.

Invasion by state $u = 1$: mathematical description

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Origin of the $2t$

- 0 "most unstable value" of $\dot{u} = u - u^2$.
  $\Longrightarrow$ Dynamics driven by small values of u.
- Linear equation around $u = 0$: $w_t - w_{xx} - w = 0$.

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Epidemiological relevance questionnable, but interesting mathematical issues

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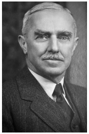
Modelling of spread of an epidemic needs, at the very least:

- A contamination mechanism
- A diffusion process.

Contamination: Kermack-McKendrick model (1927)



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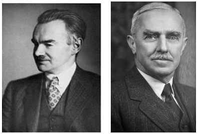

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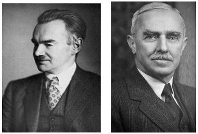

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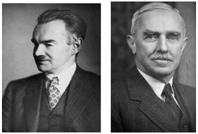

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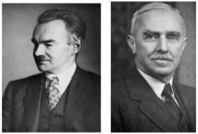

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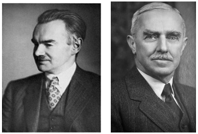

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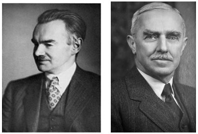

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  - Update of susceptibles: $\dot{S} = -\mathcal{I}(t,0)$.

At $t = 0$, injection of small amount of infected $I_0(i)$.

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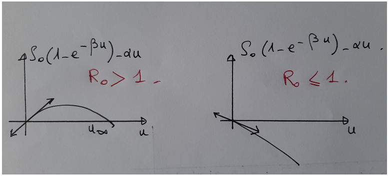
Define $R_0 = S_0\beta/\gamma$.



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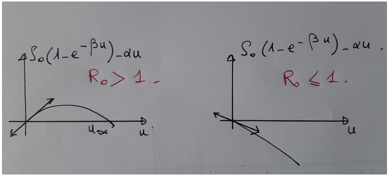

- $R_0 \leq 1$: $u(t) \to u_\infty(I_0)$ small (extinction).
- $R_0 > 1$: $u(t) \to u_\infty(I_0) > u_\infty$. Susceptibles $\to S_0 e^{-\beta u_\infty(I_0)}$.

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Analysis of Kermack-McKendrick's model

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Remark. Def. of R_0 consistent w. β and γ constant.

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— Fisher-KPP type structure: linearised equation. around 0

$$v_t + S_0\beta(v - K * v) = \gamma(R_0 - 1)v.$$

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Diffusion mechanism in Kermack-McKendrick model

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From then on we will consider the Cauchy Problem, without RHS $\mathcal{I}(0,x,i)$.

What to expect, known results

Kendall model

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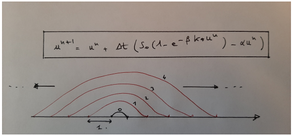

$$\lim_{t \to +\infty} u(t,x) = u_\infty, \; S_0(1 - e^{-\beta u_\infty}) - \gamma u_\infty = 0. \text{ (Kendall, 1957)}$$

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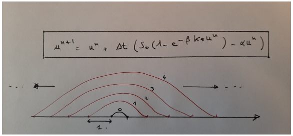

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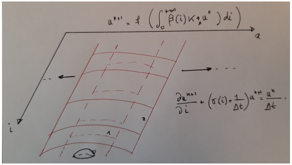
Nonlocal transport model



$$\lim_{t \to +\infty} u(t, x, i) = u_\infty(i),$$ steady solution of Kermack-McKendrick.

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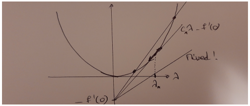

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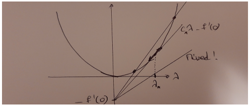

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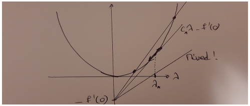

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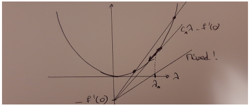

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Theorem (R., 2023). *There is x_∞ such that*

$$X(t) = c_* t - \frac{3}{2\lambda_*}\ln\, t + x_\infty + o(t).$$

Nonlocal transport model: known results

Linear wave: sol. of $\begin{cases} \partial_t u + \partial_i u = & -\gamma(i)u \qquad (t>0, x\in\mathbb{R}, i>0) \\ u(t,x,0) = & f'(0)\displaystyle\int_0^{+\infty} \beta(i)K *_x u(t,.,i)\, di. \end{cases}$

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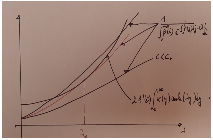

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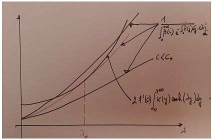

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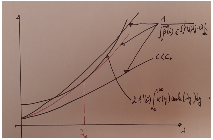

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→ Reduction to an integral equation for $u(t,x,0)$.

→ Many variants, starting point of monotone systems theory.

Sharp asymptotic behaviour

The result

Theorem (Faye, Zhang, R. 2024). $X(t) = c_* t - \dfrac{3}{2\lambda_*} \ln\, t + O(1).$

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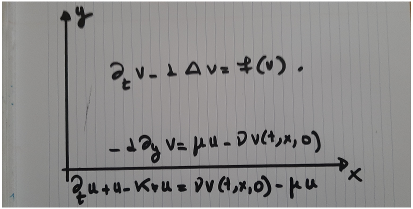
Starting point: Analogy with "Road-field model" (Berestycki, Rossi, R.)



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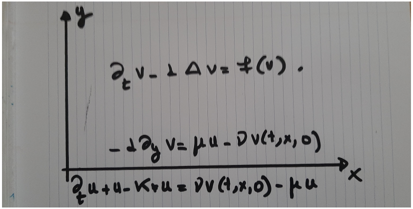

- **Idea:** view model as a road-field model instead of integral equation.
- **Common point**: global dynamics directed by line $\{y = 0\}$.
- **Main advantage:** flexibility with Cauchy Problems.
- **Main difference:** Boundary condition NOT an evolution equation.

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- **Travelling waves:** sol. of form $\phi(x-ct,i)$.

Existence iff $c \geq c_*$. *Also,* $\phi_{c_*}(x,i) \underset{x\to+\infty}{\sim} \pi_*(i) x e * -\lambda_* x.$

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— Choose $\sigma(t)$ s.t. $u(t, x = c_* t + t^\delta, 0) \underset{\leq}{\geq} \phi_{c_*}(t^\delta, 0)$.

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Work in progress

— Location of $X(t)$ with precision $o_{t \to +\infty}(1)$,

— original Kermack-McKendrick model and study of $\mathcal{I}(t, x, i)$.

Thank you!