High order cubature for iterated function system

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high order	$\rightarrow$	fast convergence
Iterated Function System (IFS)	$\rightarrow$	fractals (multiscale domain)

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high order $\rightarrow$ fast convergence

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Applications: Fractal antenna engineering

Ezhumalai, Ganesan, Balasubramaniyan (2021)

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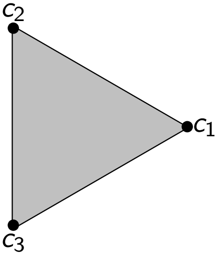
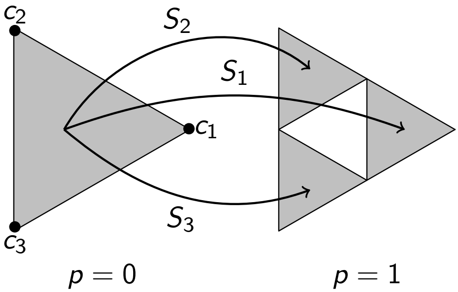
Fractal as Iterated Function System (IFS) attractor



$p = 0$

Fractal as Iterated Function System (IFS) attractor



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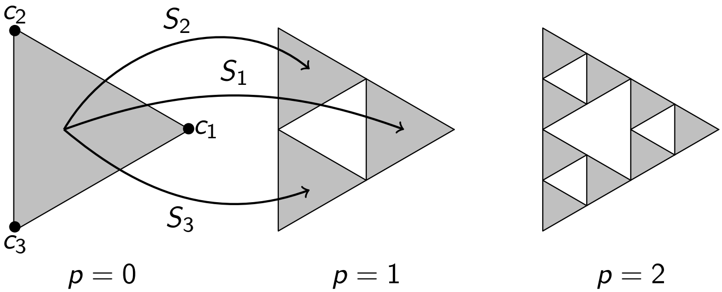

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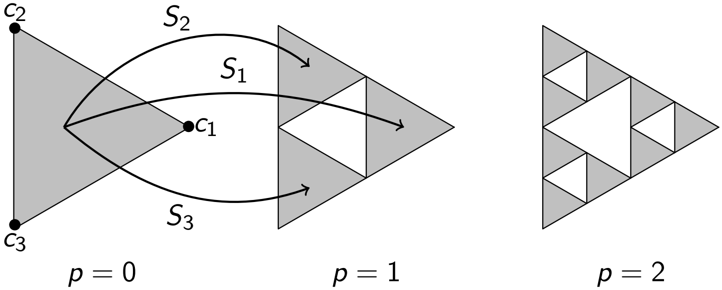

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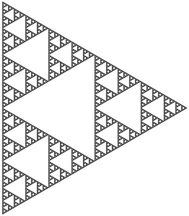

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Fractal as Iterated Function System (IFS) attractor

Iterated Functions System $\mathrm{IFS} = \{S_\ell \colon \mathbb{R}^n \to \mathbb{R}^n : \ell = 1, \dots, L\}$ where

- S_ℓ are **affine** and **contractive** ($\rho_\ell < 1$):

$$\|S_\ell(x) - S_\ell(y)\| \leq \rho_\ell \|x - y\|, \qquad \forall x, y \in \mathbb{R}^n.$$

Thm. There exists a **unique** non-empty compact set $\Gamma \subset \mathbb{R}^n$ s.t.

$$\Gamma = \mathscr{H}(\Gamma) := \bigcup_{\ell=1}^{L} S_\ell(\Gamma).$$

For any non-empty compact set F:

- $\mathscr{H}^p(F) \to \Gamma$. (for the Hausdorff distance)
- If $S_1(F), \dots, S_L(F) \subset F$, then $\Gamma = \bigcap_{p \geq 0} \mathscr{H}^p(F)$. ($\mathscr{H}^p(F)$ pre-fractal)

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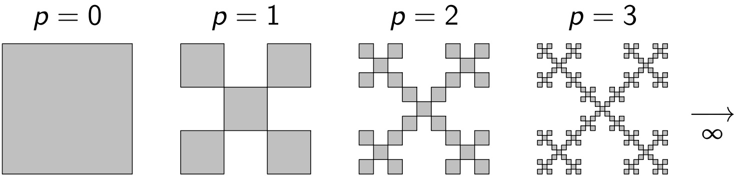
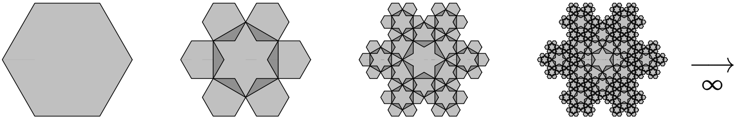
More examples

Cantor set

Vicsek



Koch snowflake



Hausdorff measure

$$\mathcal{L}^1(E) = \alpha(1) \lim_{\delta \to 0} \inf \left\{ \sum_{j \geq 1} r_j^1 \;\middle|\; E \subset \bigcup_{j \geq 1} B(x_j, r_j) \text{ and } r_j < \delta \right\}$$

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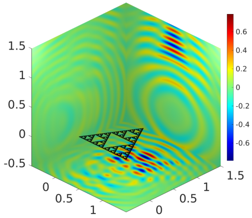
Def. $\exists! d = \dim_{\mathcal{H}} E$ such that $\mathcal{H}^{<d}(E) = +\infty$ and $\mathcal{H}^{>d}(E) = 0$.

Applications: Sound-soft scattering by a fractal screen



Gibbs, Hewett, Major (2023)

Given: k wavenumber, u^{in} incident field

Find: the scatter field u^{sc} such that

$$\text{(PDE)} \quad \begin{cases} -\Delta u^{\text{sc}} - k^2 u^{\text{sc}} = 0 & \text{in } \mathbb{R}^n \setminus \Gamma \\ u^{\text{sc}} = -u^{\text{in}} & \text{on } \Gamma \\ u^{\text{sc}} \text{ outgoing at infinity} & \end{cases}$$

We have $u^{\text{sc}}(x) = \int_\Gamma G(x,y)\phi(y)\,\mathrm{d}\mathcal{H}^d_y$, $x \in \mathbb{R}^n \setminus \Gamma$, where G fund. sol. and

$$\text{(IE)} \quad \int_\Gamma G(x,y)\phi(y)\,\mathrm{d}\mathcal{H}^d_y = u^{\text{in}}(x), \quad x \in \Gamma$$

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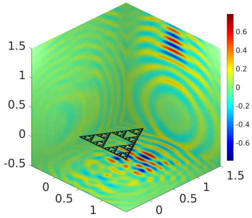
Chandler-Wilde, Hewett (2018), Caetano, Chandler-Wilde, Gibbs, Hewett, Moiola (2024)



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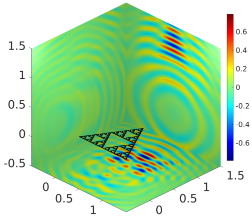

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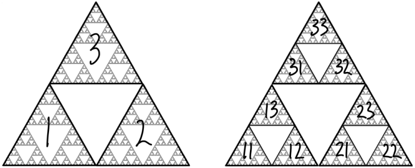
where $\Gamma_{\boldsymbol{w}} = S_{w_1} \circ \cdots \circ S_{w_p}(\Gamma)$



$$\mathbb{A}\Phi = U^{\mathrm{in}}$$

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Goal: numerically compute $\int_\Gamma f(x)\,\mathrm{d}\mathcal{H}^d_x$, for smooth f.

Gibbs, Hewett, Moiola (2023), Gibbs, Hewett, Major (2023)

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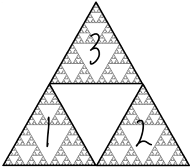

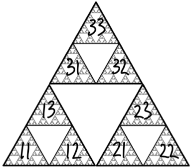

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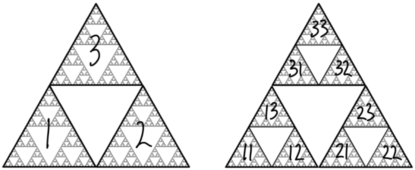

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Gibbs, Hewett, Moiola (2023), Gibbs, Hewett, Major (2023)

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Self-similar set and invariance property

- $S_\ell : x \mapsto \rho_\ell \boldsymbol{T}_\ell x + b_\ell$ $\qquad \|S_\ell(x) - S_\ell(y)\| = \rho_\ell \|x - y\|$
  - $0 < \rho_\ell < 1$.
  - $\boldsymbol{T}_\ell$ is an orthogonal matrix.
  - $b_\ell \in \mathbb{R}^n$.

Thm. Let IFS $\{S_\ell : \ell = 1, \dots, L\}$ + OSC, we have

$$\rho_1^d + \cdots + \rho_L^d = 1 \qquad \text{and} \qquad 0 < \mathcal{H}^d(\Gamma) < +\infty.$$

Rem. All examples in this talk satisfy the OSC.

Cor. For $f : \Gamma \to \mathbb{C}$, $\displaystyle\int_\Gamma f(x)\, \mathrm{d}\mathcal{H}_x^d = \sum_{1 \le \ell \le L} \rho_\ell^d \int_\Gamma f \circ S_\ell(x)\, \mathrm{d}\mathcal{H}_x^d.$

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Previous works

Goal: For $f : \Omega \supset \Gamma \to \mathbb{C}$ smooth, compute $\int_\Gamma f(x) \mathrm{d}\mu$. $\left[\mathrm{d}\mu = \frac{\mathrm{d}\mathcal{H}^d}{\mathcal{H}^d(\Gamma)} \right]$

Idea: $\int_\Gamma f(x)\, \mathrm{d}\mu \approx \sum_{i=1}^{N} w_i f(x_i)$

Known results:

- For $\Gamma \subset \mathbb{R}$: Mantica (1996)
  - Gauss rules based on orthogonal polynomials
  - high order / only works on dimension 1
- For generic IFS: Forte, Mendivil, Vrscay (1998)
  - Chaos game rules
  - convergence independent of d / stochastic + slow convergence $N^{-1/2}$
- For IFS $= \{S_\ell(x) = \boldsymbol{A}_\ell x + b_\ell\}$: Gibbs, Hewett, Moiola (2023)
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Principe: Given

- a space $\mathcal{P}$ of polynomials;
- a "good" set of points $\{x_i\}_{i=1}^N$ such that $\exists! \mathscr{L}_j \in \mathcal{P}$ Lagrange polynomials and $\mathscr{L}_j(x_i) = \delta_{i,j}$. ($\{x_i\}$ is $\mathcal{P}$-unisolvant)

Example: For $\mathcal{P} = \mathbb{Q}_k := \mathrm{span}\{X_1^{\alpha_1} X_2^{\alpha_2} \,|\, \alpha_1, \alpha_2 \leq k\}$ and $\{x_i\}$ are tensor product of Gauss-Legendre / Chebyshev points.

▶ From $f(x) \approx \sum_{i=1}^N f(x_i)\mathscr{L}_i(x)$ deduce $\{w_i\}$ s.t. $\int_\Gamma f(x)\,\mathrm{d}\mu \approx \sum_{i=1}^N w_i f(x_i)$.

Imposing that the cubature formula is exact for $f \in \mathcal{P}$ gives

$$\int_\Gamma f(x)\,\mathrm{d}\mu = \sum_{i=1}^N f(x_i) w_i, \qquad \left[w_i = \int_\Gamma \mathscr{L}_i(x)\,\mathrm{d}\mu\right].$$

Cubature

Goal: For $f : \Omega \supset \Gamma \to \mathbb{C}$ smooth, compute $\int_\Gamma f(x) \mathrm{d}\mu$. $\left[\mathrm{d}\mu = \frac{\mathrm{d}\mathcal{H}^d}{\mathcal{H}^d(\Gamma)}\right]$

Principe: Given

- a space $\mathcal{P}$ of polynomials;
- a "good" set of points $\{x_i\}_{i=1}^N$ such that $\exists! \mathscr{L}_j \in \mathcal{P}$ Lagrange polynomials and $\mathscr{L}_j(x_i) = \delta_{i,j}$. ($\{x_i\}$ is $\mathcal{P}$-unisolvant)

Example: For $\mathcal{P} = \mathbb{Q}_k := \operatorname{span}\{X_1^{\alpha_1} X_2^{\alpha_2} \mid \alpha_1, \alpha_2 \leq k\}$ and $\{x_i\}$ are tensor product of Gauss-Legendre / Chebyshev points.

▶ From $f(x) \approx \sum_{i=1}^N f(x_i)\mathscr{L}_i(x)$ deduce $\{w_i\}$ s.t. $\int_\Gamma f(x)\,\mathrm{d}\mu \approx \sum_{i=1}^N w_i f(x_i)$.

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Goal: computing $w_i = \int_\Gamma \mathscr{L}_i(x)\,\mathrm{d}\mu$

Rem. There exists a recursive algorithm (on the total degree) for computing the integral of monomials. Strichartz (2000)

Def. A polynomial space $\mathcal{P}$ is *$\mathscr{S}$-invariant* if

$$p \circ S_1, \dots, p \circ S_L \in \mathcal{P}, \qquad \forall p \in \mathcal{P}.$$

Examples:

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- $\mathbb{Q}_k := \mathrm{span}\{X_1^{\alpha_1} X_2^{\alpha_2} \,|\, \alpha_1, \alpha_2 \leq k\}$ is $\mathscr{S}$-invariant if $\boldsymbol{T}_\ell = \mathrm{diag}(\pm 1 \dots)$.

Weights computation for $\mathscr{S}$-invariant spaces

Goal: computing $w_i = \int_\Gamma \mathscr{L}_i(x)\,\mathrm{d}\mu$

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$$w_i = \int_\Gamma \mathscr{L}_i(x)\,\mathrm{d}\mu = \sum_{1\le\ell\le L} \rho_\ell^d \int_\Gamma \mathscr{L}_i \circ S_\ell(x)\,\mathrm{d}\mu$$

$$= \sum_{1\le\ell\le L} \rho_\ell^d \int_\Gamma \sum_{1\le j\le N} \mathscr{L}_i \circ S_\ell(x_j)\mathscr{L}_j(x)\,\mathrm{d}\mu$$

$$= \sum_{1\le j\le N} \left[\sum_{1\le\ell\le L} \rho_\ell^d \mathscr{L}_i \circ S_\ell(x_j)\right] \underbrace{\int_\Gamma \mathscr{L}_j(x)\,\mathrm{d}\mu}_{=w_j}$$

We get $\boxed{\mathbf{S}w = w}$ where $\mathbf{S}_{i,j} = \sum_{1\le\ell\le L} \rho_\ell^d \mathscr{L}_i \circ S_\ell(x_j)$.

Lem. 1 is a simple eigenvalue of $\mathbf{S}$ and $|\mathrm{Sp}(\mathbf{S}) \setminus \{1\}| \le \max_\ell \rho_\ell < 1$.

Joly, Kachanovska, Moitier (2024)

Therefore, $\mathbf{S}w = w$ with $w_1 + \cdots + w_N = 1$ has a unique solution.

Numeric: w_i computed using a power iteration method on $\mathbf{S}$.

Weights computation for $\mathscr{S}$-invariant spaces

$$
\begin{aligned}
w_i = \int_\Gamma \mathscr{L}_i(x)\,\mathrm{d}\mu &= \sum_{1\leq \ell\leq L} \rho_\ell^d \int_\Gamma \mathscr{L}_i \circ S_\ell(x)\,\mathrm{d}\mu \\
&= \sum_{1\leq \ell\leq L} \rho_\ell^d \int_\Gamma \sum_{1\leq j\leq N} \mathscr{L}_i \circ S_\ell(x_j) \mathscr{L}_j(x)\,\mathrm{d}\mu \\
&= \sum_{1\leq j\leq N} \left[\sum_{1\leq \ell\leq L} \rho_\ell^d \mathscr{L}_i \circ S_\ell(x_j) \right] \underbrace{\int_\Gamma \mathscr{L}_j(x)\,\mathrm{d}\mu}_{=w_j}
\end{aligned}
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Example. In 1d, $\Lambda_N^{\text{equispaced}} \sim \frac{2^{N+1}}{\mathrm{e}N\log N}$ and $\Lambda_N^{\text{Chebyshev}} \sim \frac{2}{\pi}\log(N+1)$.

Cantor set ($S_1(x) = \rho x$ and $S_2(x) = \rho x + 1 - \rho$)

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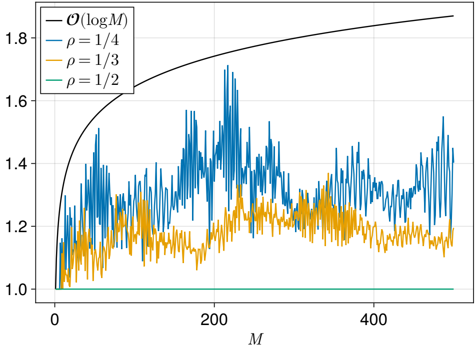
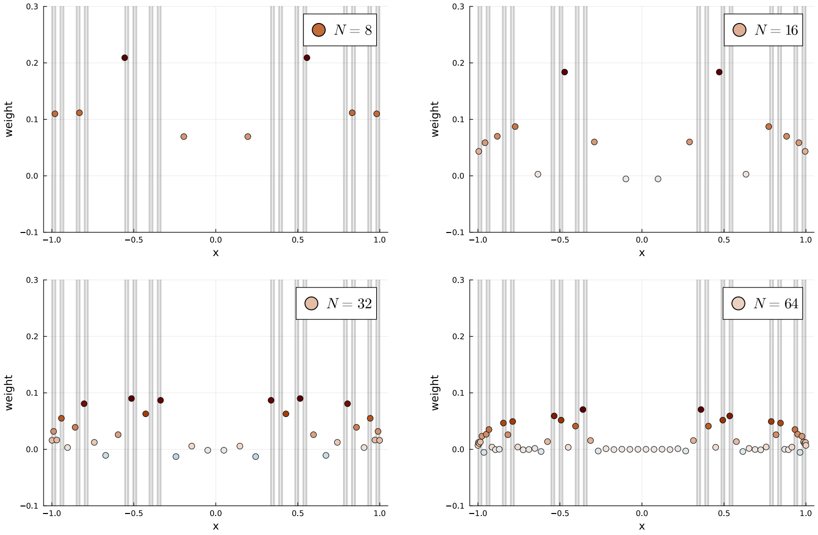


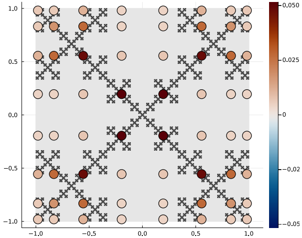
Cantor set ($S_0(x) = x/3$ and $S_0(x) = x/3 + 2/3$)



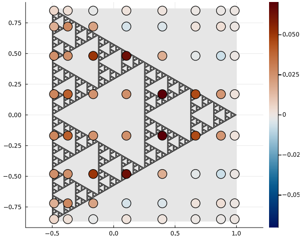
Click for video.

$\mathscr{S}$-invariant case: Numeric on polynomials

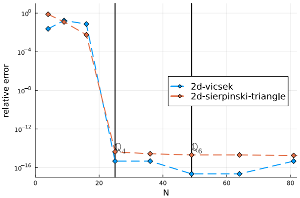
Vicsek



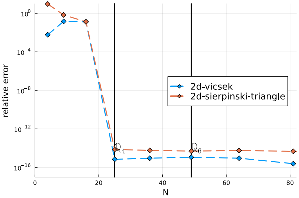
Sierpiński triangle



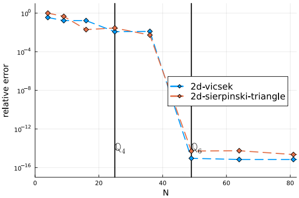
The points (t_i, t_j) where $\{t_i\}$ Chebyshev points.



$x \mapsto x_1^4$



$x \mapsto x_1^4 x_2^2$



$x \mapsto x_1^6$

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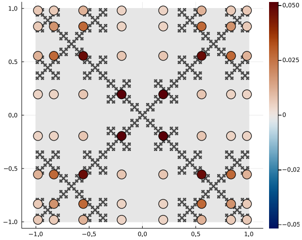

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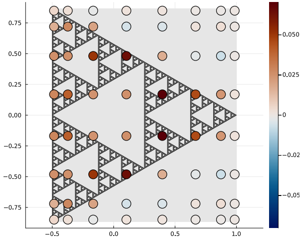

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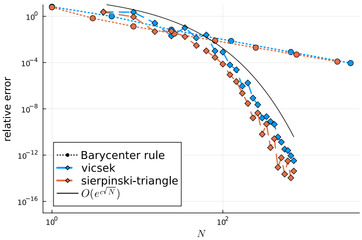

Error:

$$\mathcal{O}(e^{-c\sqrt{N}})$$

Non $\mathscr{S}$-invariant case: Description

Hyp: $\mathcal{P} = \mathbb{Q}_k$ and $\boldsymbol{T}_\ell$ arbitrary. [non $\mathscr{S}$-invariant]

- For $w_i = \int_\Gamma \mathscr{L}_i(x)\,\mathrm{d}\mu$, we have $\mathbf{S}w \neq w$.
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Conjecture. 1 is a simple eigenvalue of $\mathbf{S}$ and $|\mathrm{Sp}(\mathbf{S}) \setminus \{1\}| \leq \max_\ell \rho_\ell$.

▶ Proof when $\rho_\ell < c(N,k)$. ▶ Verified numerically.

Thm. Any $\mathscr{S}$-inv. subspace $\underbrace{\mathcal{P}^{\mathscr{S}}}_{``\mathbb{P}_k"} \subset \underbrace{\mathcal{P}}_{``\mathbb{Q}_k"}$ is exactly int. by $\sum_i \widetilde{w}_i f(x_i)$.

Joly, Kachanovska, Moitier (2024)

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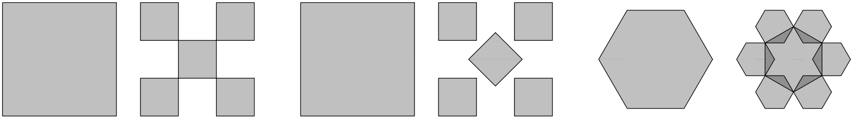
Joly, Kachanovska, Moitier (2024)

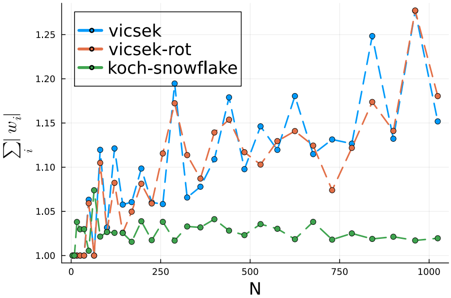


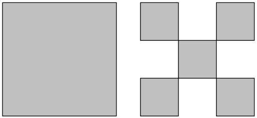
Vicsek | Vicsek with rotation | Koch snowflake

The points (t_i, t_j) where $\{t_i\}$ Chebyshev points.

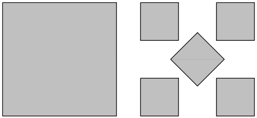
No theoretical bound on

$$\sum_{i=1}^{N} |w_i|$$

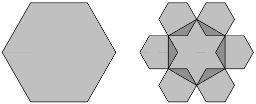




Vicsek



Vicsek with rotation



Koch snowflake

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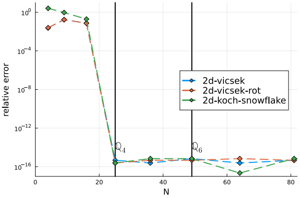
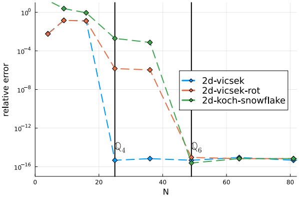


$x \mapsto x_1^4$



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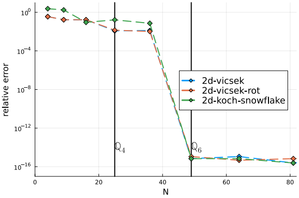

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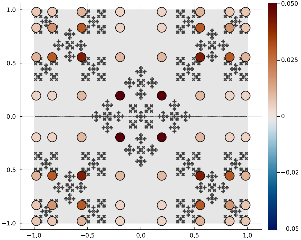
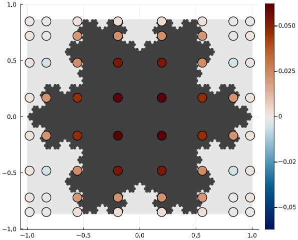
Non $\mathscr{S}$-invariant case: Numerics

Vicsek with rotation



Koch snowflake



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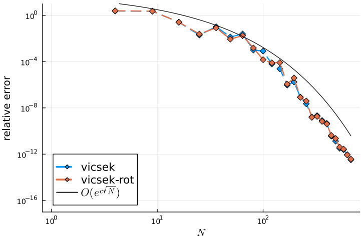
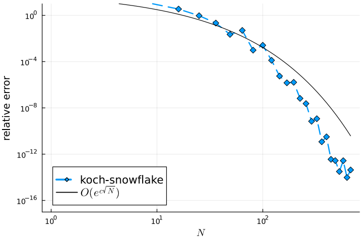
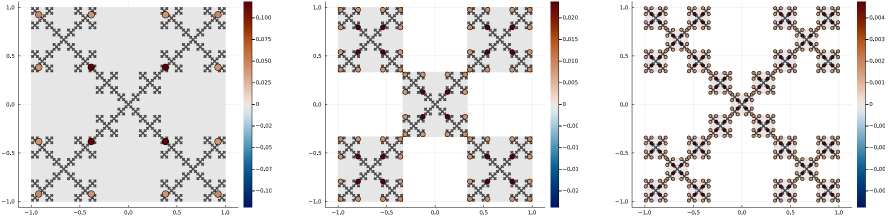
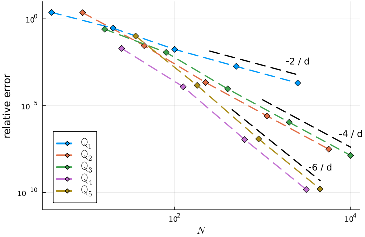




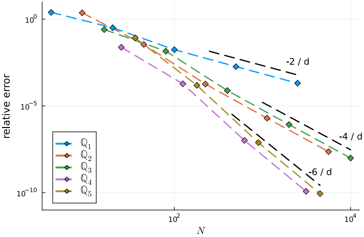
Fix N and refine the cubature using S_ℓ.



Vicsek



Vicsek with rotation



$$f(x) = \frac{e^{ik\|x-x_0\|}}{\|x-x_0\|}$$

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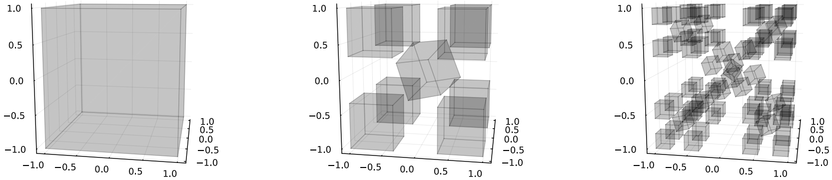
"3d Vicsek" with rotation

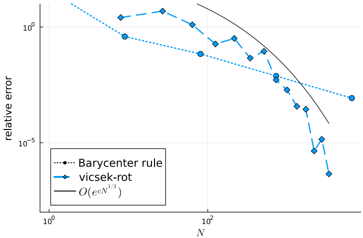


$$f(x) = \frac{e^{ik\|x-x_0\|}}{\|x-x_0\|}$$

$k = 5$

$x_0 = (2, 0.5, 1)$

p-version



h-version

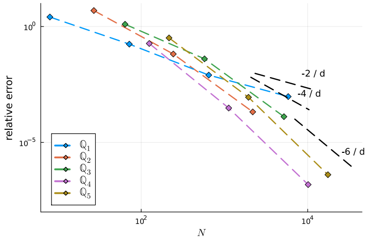


Table of Contents

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- We have constructed high order cubature (h-version and p-version) for $\displaystyle\int_\Gamma f(x)\,\mathrm{d}\mu$ where Γ is an IFS attractor.

Remark:

- Works with other invariant measure than Hausdorff.
- Works with self-affine set.

Perspective:

- Incorporate this cubature in the full BEM case.
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Thank you for your attention

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Thank you for your attention

Bibliography

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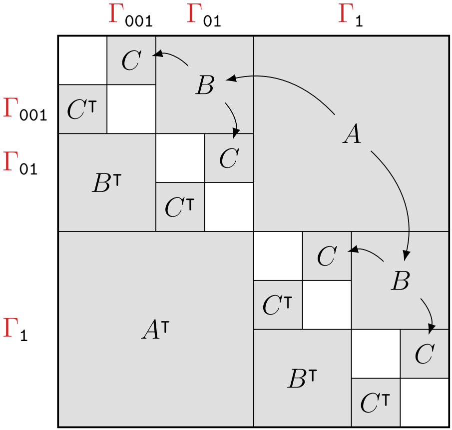
Cantor set: $\mathcal{H}$-matrix

$$\mathbb{A}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \iint_{\Gamma_{\boldsymbol{\alpha}} \times \Gamma_{\boldsymbol{\beta}}} \|x - y\|^{-1} \, \mathrm{d}\mathcal{H}^d_y \, \mathrm{d}\mathcal{H}^d_x$$



We have

$$B = \rho^{2d-1}(A_{0,0} + A_{0,1} + A_{1,0} + A_{1,1})$$

where

$$A = \begin{pmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{pmatrix}$$

Lem. If A is r-low-rank than B is r-low-rank.

Proof. If $A \approx X^\intercal Y$ with $\|A - X^\intercal Y\| < \varepsilon$, we note $X = \begin{pmatrix} X_0 \\ X_1 \end{pmatrix}$, $Y = \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix}$ then $B \approx \rho^{2d-1}(X_0 + X_1)^\intercal (Y_0 + Y_1)$ width

$$\|B - \rho^{2d-1}(X_0 + X_1)^\intercal (Y_0 + Y_1)\| < \rho^{2d-1} L\varepsilon.$$