Joint inviscid-incompressible limit for tissue growth models: from Brinkman's law to Hele-Shaw

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Joint work (in progress) with **Matt Jacobs** and **Inwon Kim**

Rencontres normandes sur les aspects théoriques et numériques des EDP

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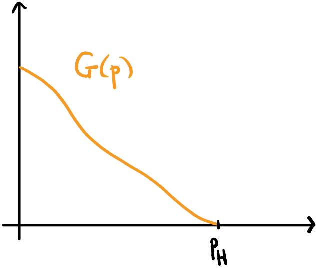

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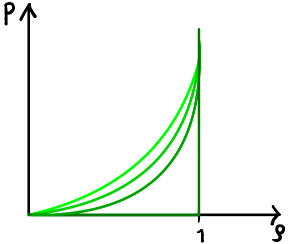

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We will consider: $\mathrm{v} = -\nabla W$, with $-\nu \Delta W + W = p$.

Singular limits

Brinkman's law

$$\begin{aligned} \partial_t \varrho - \nabla \cdot (\varrho \nabla W) &= \varrho G(p), \\ -\nu \Delta W + W &= p, \\ p &= \varrho^{\gamma}. \end{aligned}$$

$$\xrightarrow{\ \nu \to 0\ }$$

Darcy's law/PME

$$\begin{aligned} \partial_t \varrho - \nabla \cdot (\varrho \nabla p) &= \varrho G(p), \\ p &= \varrho^{\gamma}. \end{aligned}$$

$$\Big\downarrow \ \gamma \to \infty \qquad\qquad \Big\downarrow \ \gamma \to \infty$$

$$\begin{aligned} \partial_t \varrho - \nabla \cdot (\varrho \nabla W) &= \varrho G(p), \\ -\nu \Delta W + W &= p, \\ p(1 - \varrho) &= 0, \\ p(\Delta W + G(p)) &= 0. \end{aligned}$$

Incompressible Brinkman

$$\begin{aligned} \partial_t \varrho - \nabla \cdot (\varrho \nabla p) &= \varrho G(p), \\ p(1 - \varrho) &= 0, \\ p(\Delta p + G(p)) &= 0. \end{aligned}$$

Incompressible Darcy/Hele-Shaw

(Short) literature review

Incompressible limit

- $\nu = 0, \gamma \to \infty$: From PME to HS: *Perthame, Vázquez, Quirós (2014)*
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  Compactness of ∇W_γ, $W_\nu = K_\nu \star p_\nu$, $-\nu \Delta K_\nu + K_\nu = \delta_0$,
  strong compactness of p_γ through Young's measures interpretation

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Inviscid limit

- $\boldsymbol{\nu \to 0}, \gamma \geq 1$: **From compressible Brinkman to PME:** *D., Dębiec, Mandal, Schmidtchen ('23) [$\gamma = 1$], Elbar, Skrzeczkowski ('23) [$\gamma > 1$]*
  Entropy/Energy (in)equalities, strong compactness of W_ν, p_ν through pressure's equation:
  $\partial_t p = \gamma p(\Delta W + G(p)) + \nabla p \cdot \nabla W$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
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Joint limit? Inviscid for $\gamma = \infty$?

Outline

Assumptions: approximating sequence $f_\nu,\ p_\nu \in \partial f_\nu(\varrho_\nu)$

The family of energy dissipation inequalities

A priori estimates: weak compactness of $(\varrho_\nu p_\nu)_{\nu>0}$

Strong compactness of $(\nabla W_\nu)_{\nu>0}$

Family of energy functions:

- For all $\nu > 0$, $f_\nu : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a lower semi-continuous, convex function, and $f_\nu(0) = 0$,
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Example

$$f_\nu(\varrho) = \frac{\nu}{\nu+1}\varrho^{\frac{1}{\nu}+1}, \quad f_\nu'(\varrho) = \varrho^{\frac{1}{\nu}}.$$

As $\nu \to 0$ it converges to the incompressible energy

$$f_0(\varrho) = \begin{cases} 0, & \text{for } \varrho < 1 \\ +\infty, & \text{for } \varrho \geq 1. \end{cases}$$

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Initial data: $\varrho^{\mathrm{in}} \geq 0$, $\varrho^{\mathrm{in}} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $|x|^2\varrho^{\mathrm{in}} \in L^1(\mathbb{R}^d)$.

Setting

Goal:

$$\begin{aligned} \partial_t \varrho_\nu - \nabla \cdot (\varrho_\nu \nabla W_\nu) &= \varrho_\nu G(p_\nu), \\ -\nu \Delta W_\nu + W_\nu = p_\nu &\in \partial f_\nu(\varrho_\nu), \end{aligned} \qquad \nu \to 0 \qquad \begin{aligned} \partial_t \varrho - \nabla \cdot (\varrho \nabla p) &= \varrho G(p), \\ p &\in \partial f_0(\varrho) \end{aligned}$$

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$$p_\nu \varrho_\nu = f_\nu(\varrho_\nu) + f_\nu^*(p_\nu), \qquad \qquad p\varrho = f_0(\varrho) + f_0^*(p).$$

Recall: $f^*(b) = \sup_a ab - f(a)$.

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$$f_0^*(b) = \begin{cases} b, & \text{for } b > 0, \\ 0, & \text{for } b \leq 0, \end{cases} \qquad f_0^{*\prime}(b) = \begin{cases} 1, & \text{for } b > 0, \\ 0, & \text{for } b \leq 0. \end{cases}$$

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A priori estimates: $\varrho_\nu, p_\nu \in L^\infty(0,T; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ uniformly $\nu > 0$, thus

$$\varrho_\nu \rightharpoonup \varrho, \quad p_\nu \rightharpoonup p.$$

Moreover

$$W_\nu = K_\nu \star p_\nu \rightharpoonup p.$$

Energy dissipation inequalities

Theorem: Energy dissipation inequality.

Let $z : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a convex function, and $h_\nu : \mathbb{R} \to \mathbb{R}$ such that

$$h_\nu'(b) = \int_{b_0}^{b} \frac{z''(s)}{f_\nu^{*\prime}(s)} ds, \text{ for some } b_0 \in \mathbb{R}.$$

Then

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but use the regularised equation: $\partial_t \varrho - \varepsilon\Delta\varrho - \nabla\cdot(\varrho\nabla W) = \varrho G(p)$.

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Internal energy dissipation

Choose: $z'' = f_\nu^{*\prime}$, then $h_\nu'(p_\nu) = p_\nu = f_\nu'(\varrho_\nu)$ and obtain

$$\int_{\mathbb{R}^d} f_\nu(\varrho_\nu)(T) + \int_0^T \int_{\mathbb{R}^d} f_\nu^{*\prime}(W_\nu)|\nabla W_\nu|^2 \leq \int_0^T \int_{\mathbb{R}^d} p_\nu \varrho_\nu G(p_\nu) + \int_{\mathbb{R}^d} f_\nu(\varrho_\nu)(0),$$

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This is the equivalent of the energy equality for **Darcy's law**

$$\int_{\mathbb{R}^d} f(\varrho)(T) + \int_0^T\int_{\mathbb{R}^d} \varrho|\nabla p|^2 = \int_0^T\int_{\mathbb{R}^d} p\varrho G(p) + \int_{\mathbb{R}^d} f(\varrho)(0).$$

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The **PME** is a W_2-**gradient flow** with respect to the internal energy

$$F(\varrho) = \int_{\mathbb{R}^d} f(\varrho)dx,$$

namely

$$\partial_t\varrho = -\nabla_{W_2}F(\varrho) = \nabla\cdot\left(\varrho\nabla\frac{\delta F(\varrho)}{\delta\varrho}\right).$$

Indeed $\dfrac{\delta F(\varrho)}{\delta\varrho} = f'(\varrho) = p$, hence

$$\partial_t\varrho - \nabla\cdot(\varrho\nabla p) = 0.$$

Entropy dissipation

Choose: $z'' = f_\nu^{*\prime\prime}$, then $h_\nu'(p_\nu) = \ln(f_\nu^{*\prime}(p_\nu)) = \ln \varrho_\nu$ and obtain

$$\mathcal{H}(\varrho_\nu)(T) + \int_0^T \int_{\mathbb{R}^d} f_\nu^{*\prime\prime}(W_\nu)|\nabla W_\nu|^2 \leq \int_0^T \int_{\mathbb{R}^d} \ln \varrho_\nu \varrho_\nu G(p_\nu) + \mathcal{H}(\varrho_\nu)(0)$$

where

$$\mathcal{H}(\varrho) = \int_{\mathbb{R}^d} \varrho \ln \varrho - \varrho.$$

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Remark. $\varrho_\nu \ln \varrho_\nu \in L^1(0,T; L^1(\mathbb{R}^d))$ uniformly in $\nu \geq 0$ because the second moment control propagates in time

$$\sup_{\nu>0} \sup_{t\in[0,T]} \int_{\mathbb{R}^d} |x|^2 \varrho_\nu < +\infty.$$

Choose: $z'' = 1$, then $h'_\nu(p_\nu) = \int \frac{1}{f^{*\prime}_\nu(s)} ds$ and obtain

$$\int_{\mathbb{R}^d} (\varrho_\nu h'_\nu(p_\nu) - z'(p_\nu))(T) + \int_0^T \int_{\mathbb{R}^d} \nu |\Delta W_\nu|^2 + \int_0^T \int_{\mathbb{R}^d} |\nabla W_\nu|^2$$
$$\leq \int_0^T \int_{\mathbb{R}^d} h'_\nu(p_\nu) \varrho_\nu G(p_\nu) + \int_{\mathbb{R}^d} (\varrho_\nu h'_\nu(p_\nu) - z'(p_\nu))(0),$$

Energy dissipation inequality

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$$\leq \int_0^T \int_{\mathbb{R}^d} h'_\nu(p_\nu)\varrho_\nu G(p_\nu) + \int_{\mathbb{R}^d} (\varrho_\nu h'_\nu(p_\nu) - z'(p_\nu))(0),$$

because

$$\begin{aligned}
\iint h'_\nu(p_\nu)\nabla p_\nu \cdot \varrho_\nu \nabla W_\nu &= -\iint z'(p_\nu)\Delta W_\nu \\
&= -\iint p_\nu \Delta W_\nu \\
&= -\iint (-\nu\Delta W_\nu + W_\nu)\Delta W_\nu \\
&= \iint \nu|\Delta W_\nu|^2 + |\nabla W_\nu|^2.
\end{aligned}$$

Compactness of the product

Lemma

The following holds uniformly in $\nu > 0$

- $\sqrt{\nu}\Delta W_\nu \in L^2(0,T;L^2(\mathbb{R}^d))$,
- $W_\nu \in L^2(0,T;H^1(\mathbb{R}^d))$,
- $\partial_t \varrho_\nu \in L^2(0,T;H^{-1}(\mathbb{R}^d))$.

Moreover

$$\varrho_\nu p_\nu \rightharpoonup \varrho p \quad \text{weakly in } L^2(0,T;L^2(\mathbb{R}^d)).$$

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Proof. By Brinkman's law

$$\boxed{p_\nu - W_\nu = -\nu\Delta W_\nu \to 0.}$$

Hence

$$\iint p_\nu \varrho_\nu \varphi = \iint (W_\nu \varrho_\nu - \nu\Delta W_\nu \varrho_\nu)\varphi \to \iint p\varrho\varphi,$$

and since ϱ_ν, p_ν are uniformly bounded in any L^p, we conclude.

Convergence of $f_\nu^*(W_\nu)$

Lemma

Up to a subsequence

$$f_\nu(\varrho_\nu) \rightharpoonup f_0(\varrho), \quad f_\nu^*(p_\nu) \rightharpoonup f_0^*(p), \quad f_\nu^*(W_\nu) \rightharpoonup f_0^*(p).$$

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Proof. We know

$$f_\nu(\varrho_\nu) + f_\nu^*(p_\nu) = \varrho_\nu p_\nu \rightharpoonup \varrho p \leq f_0(\varrho) + f_0^*(p).$$

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$$f_\nu(\varrho_\nu) \rightharpoonup f_0(\varrho), \qquad f_\nu^*(p_\nu) \rightharpoonup f_0^*(p).$$

Finally, we compute

$$\begin{aligned} |f_\nu^*(p_\nu) - f_\nu^*(W_\nu)| &\le f_\nu^{*\prime}(\max(p_\nu, W_\nu))|p_\nu - W_\nu| \\ &\le \|\varrho_\nu\|_\infty |p_\nu - W_\nu| \to 0. \end{aligned}$$

Compactness of ∇W_ν

We know: $\varrho_\nu \rightharpoonup \varrho,\ p_\nu \rightharpoonup p\ \text{ in } L^2L^2,\ W_\nu \rightharpoonup p\ \text{ in } L^2H^1,$

and $\varrho_\nu \nabla W_\nu \rightharpoonup m,\ \varrho_\nu G(p_\nu) \rightharpoonup R\ \text{ in } L^2L^2.$

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Then, we have $$\partial_t \varrho - \nabla \cdot m = R.$$

We need to prove: $$m = \varrho \nabla p,$$

(strong compactness of ∇W_ν and p_ν will follow).

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Idea: EDI formulation of gradient flows (Sandier-Serfaty).

Problem: The Brinkman equation is not a gradient flow!

Compactness of ∇W_ν: proof

We show that

$$(1) \qquad \iint \frac{|m|^2}{2\varrho} \le \liminf_{\nu\to 0} \frac{1}{2} \iint f_\nu^{*\prime}(W_\nu)|\nabla W_\nu|^2$$

$$(2) \qquad \iint \frac{\varrho|\nabla p|^2}{2} \le \liminf_{\nu\to 0} \frac{1}{2} \iint f_\nu^{*\prime}(W_\nu)|\nabla W_\nu|^2$$

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To prove (3) we compare

$$\int_{\mathbb{R}^d} f_\nu(\varrho_\nu)(T) + \int_0^T\int_{\mathbb{R}^d} f_\nu^{*\prime}(W_\nu)|\nabla W_\nu|^2 \leq \int_0^T\int_{\mathbb{R}^d} p_\nu \varrho_\nu G(p_\nu) + \int_{\mathbb{R}^d} f_\nu(\varrho_\nu)(0)$$

and

$$\int_{\mathbb{R}^d} f_0(\varrho)(T) + \int_0^T\int_{\mathbb{R}^d} m \cdot \nabla p = \int_0^T\int_{\mathbb{R}^d} pR + \int_{\mathbb{R}^d} f_0(\varrho)(0)$$

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In [D., Dębiec, Mandal, Schmidtchen, SIMA 24], we had $p_\nu = f'(\varrho_\nu) = \varrho_\nu$ and the **entropy** dissipation gives $\iint |\nabla W_\nu|^2$.

Compactness of ∇W_ν: proof

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Main idea: estimate how "far" $f_\nu^{*\prime}(W_\nu)$ is from ϱ_ν.

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If we had $\boldsymbol{\varrho_\nu}$ (take ζ smooth vector field)

$$\begin{aligned} \frac{1}{2} \iint \varrho_\nu |\nabla W_\nu|^2 &\geq \iint \varrho_\nu \nabla W_\nu \cdot \zeta - \frac{1}{2} \varrho_\nu |\zeta|^2 \\ &\to \iint m \cdot \zeta - \frac{1}{2} \varrho |\zeta|^2. \end{aligned}$$

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Need to show

$$\limsup_{\nu\to 0} \frac{1}{2}\iint (\varrho_\nu - f_\nu^{*\prime}(W_\nu))|\nabla W_\nu|^2 \le 0.$$

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$$\limsup_{\nu\to 0} \frac{1}{2}\iint (\varrho_\nu - f_\nu^{*\prime}(W_\nu))|\nabla W_\nu|^2 \le 0.$$

Remark. For $f_\nu = f_0$ it is trivial: $W_\nu > 0$ and $f_\nu^*(b) = (b)_+$, while $\varrho_\nu \le 1$.

Compactness of ∇W_ν: proof

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Need to show

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Idea: estimate ϱ_ν from **above** $\varrho_\nu \le \dfrac{f_\nu^*(p_\nu + \delta) - f_\nu^*(p_\nu)}{\delta}$.

Compactness of ∇W_ν: proof

$$(2) \qquad \iint \frac{\varrho|\nabla p|^2}{2} \leq \liminf_{\nu\to 0} \frac{1}{2} \iint f_\nu^{*\prime}(W_\nu)|\nabla W_\nu|^2.$$

Compactness of ∇W_ν: proof

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We use Young's inequality and integration by parts

$$\begin{aligned} \frac{1}{2} \iint f_\nu^{*\prime}(W_\nu)|\nabla W_\nu|^2 &\geq \iint f_\nu^{*\prime}(W_\nu)\nabla W_\nu \cdot \zeta - \frac{1}{2} \iint f_\nu^{*\prime}(W_\nu)|\zeta|^2 \\ &= -\iint f_\nu^{*}(W_\nu)\nabla \cdot \zeta - \frac{1}{2} \iint f_\nu^{*\prime}(W_\nu)|\zeta|^2 \end{aligned}$$

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Use the **inf-convolution** $f_{\nu,\delta}^*(b) := \inf_c f_\nu^*(c) + \frac{1}{2\delta}|c-b|^2$ which satisfies

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Compactness of ∇W_ν: proof

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- $f_{\nu,\delta}^*$ is $\frac{1}{\delta}$-$W^{2,\infty}$, and $|p_\nu - W_\nu| \to 0$.

Conclusion

$$
\begin{tikzcd}
\begin{array}{r} \partial_t \varrho - \nabla \cdot (\varrho \nabla W) = \varrho G(p), \\ -\nu \Delta W + W = p, \\ p = \varrho^{\gamma}. \end{array} \arrow[r, "\nu \to 0"] \arrow[d, "\gamma \to \infty"] \arrow[dr, "\boldsymbol{\nu \to 0}"', "\boldsymbol{\gamma \to \infty}"] & \begin{array}{r} \partial_t \varrho - \nabla \cdot (\varrho \nabla p) = \varrho G(p), \\ p = \varrho^{\gamma}. \end{array} \arrow[d, "\gamma \to \infty"] \\
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Thank you for your attention!

Compactness of ∇W_ν: proof

Main idea: estimate how "far" $f_\nu^{*\prime}$ is from ϱ_ν.

$$(1) \quad \iint \frac{|m|^2}{2\varrho} \leq \liminf_{\nu \to 0} \frac{1}{2} \iint f_\nu^{*\prime}(W_\nu)|\nabla W_\nu|^2.$$

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If we had ϱ_ν

$$\begin{aligned}
\frac{1}{2}\iint \varrho_\nu |\nabla W_\nu|^2 &\geq \iint \varrho_\nu q(\nabla W_\nu) \\
&\geq \iint \varrho_\nu \zeta \cdot \nabla W_\nu - \varrho_\nu q^*(\zeta) \\
&\to \iint \zeta \cdot m - \varrho q^*(\zeta) \\
&\geq \iint \varrho q\left(\frac{m}{\varrho}\right),
\end{aligned}$$

take $q \to |x|^2/2$ and conclude.

Compactness of ∇W_ν: proof

Lemma

$$\limsup_{\nu \to 0} \frac{1}{2} \iint (\varrho_\nu - f_\nu^{*\prime}(W_\nu)) q(\nabla W_\nu) \leq 0.$$

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Idea of the proof: estimate ϱ_ν from **above**: since $\varrho_\nu \in \partial f_\nu^*(p_\nu)$

$$\varrho_\nu \le \frac{f_\nu^*(p_\nu+\delta) - f_\nu^*(p_\nu)}{\delta}.$$

Compactness of ∇W_ν: proof

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$$\lim_{\delta\to 0}\limsup_{\nu\to 0} \frac{1}{2}\iint \left(\frac{f_\nu^*(p_\nu+\delta) - f_\nu^*(p_\nu)}{\delta} - f_\nu^{*\prime}(W_\nu)\right) q(\nabla W_\nu) \le 0$$

Compactness of ∇W_ν: proof

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$$+ \frac{1}{2} \iint \left(\frac{f_\nu^*(W_\nu + \delta) - f_\nu^*(W_\nu)}{\delta} - f_\nu^{*\prime}(W_\nu) \right) q(\nabla W_\nu) \le 0$$

Compactness of ∇W_ν: proof

For all $\delta > 0$

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since $p_\nu - W_\nu \to 0$ strongly in $L^2 L^2$.

Assume $f_\nu^{*\prime}$ **concave** on $(0, +\infty)$ (e.g. the power law). Then,

$$\frac{1}{2} \iint \left(\frac{f_\nu^*(W_\nu + \delta) - f_\nu^*(W_\nu)}{\delta} - f_\nu^{*\prime}(W_\nu) \right) q(\nabla W_\nu) \le \delta \iint f_\nu^{*\prime\prime}(W_\nu) q(\nabla W_\nu) \le C\delta,$$

with C uniform in $\nu > 0$. We used the **entropy dissipation inequality**

$$\mathcal{H}_\nu(\varrho_\nu)(T) + \int_0^T \int_{\mathbb{R}^d} f_\nu^{*\prime\prime}(W_\nu) |\nabla W_\nu|^2 \le C.$$

Compactness of ∇W_ν: proof

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Compactness of ∇W_ν: proof

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Goal:
$\lim_{\delta\to 0}\liminf_{\nu\to 0}\iint(f^*_\nu(W_\nu) - f^*_{\nu,\delta}(W_\nu))\nabla\cdot\zeta + \iint(\varrho_\nu - f^{*\prime}_{\nu,\delta}(W_\nu))|\zeta|^2 \geq 0.$

Compactness of ∇W_ν: proof

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$$\lim_{\delta\to 0}\liminf_{\nu\to 0}\iint (f_\nu^*(W_\nu)-f_{\nu,\delta}^*(W_\nu))\nabla\cdot\zeta+\iint(\varrho_\nu - f_{\nu,\delta}^{*\prime}(W_\nu))|\zeta|^2\geq 0.$$

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Compactness of ∇W_ν: proof

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We have

$$|f_\nu^*(W_\nu) - f_{\nu,\delta}^*(W_\nu)| \leq \frac{\delta}{2}|f_\nu^{*\prime}(W_\nu)|^2 \leq \frac{\delta}{2}\|\varrho_\nu\|_\infty^2,$$

and

$$\varrho_\nu - f_{\nu,\delta}^{*\prime}(W_\nu) = f_\nu^{*\prime}(p_\nu) - f_{\nu,\delta}^{*\prime}(W_\nu) \geq f_{\nu,\delta}^{*\prime}(p_\nu) - f_{\nu,\delta}^{*\prime}(W_\nu) \geq -\frac{1}{\delta}|p_\nu - W_\nu| \to 0.$$