Traffic models on a Junction: From Micro to Macro

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Abstract

Traffic flow problems on junctions (or intersections) have been the subject of abundant literature in recent years. The modeling involves scalar conservation laws with discontinuities at the junction points, or, sometimes equivalently, Hamilton-Jacobi equations with discontinuous Hamiltonians. We will present the existence and uniqueness results for these equations, then explain how to derive these continuous models (where traffic is seen as a fluid) from discrete models (describing in detail the individual behavior of vehicles). We conclude with open issues.

Contents

1 Introduction

The goal of the course is to describe recent researches which intend to derive continuous traffic law models from the microscopic behavior of vehicles. We will start from a traffic flow on a single line, where this derivation is by now well understood. We will then increasingly complicate the models to arrive to problems on junctions, for which many questions remain unanswered. For simplicity, we focus on the case of a single node: this is without loss of generality because of the finite speed of propagation which is satisfied by our models.

One of the interests of the topic is that it touches several very active domains in PDEs: conservation laws (possibly with discontinuous fluxes), Hamilton-Jacobi equation (possibly with discontinuous Hamiltonians), determinist and stochastic homogenization of conservation laws and Hamilton-Jacobi equations, hydrodynamic limits...

Let us warn the reader that traffic flow on networks has been a very active research area for at least three decades. A general overview of the rich literature on the subject is completely out of the scope of these short notes: we refer for instance to the survey papers or monographs [11, 14, 33] and the references therein. The originality of the approach developed in this survey—if any—is to emphasize the derivation of continuous laws from discrete laws.

2 Traffic flow on the line

2.1 A discrete model: follow-the-leader

The simplest model of traffic flow is the so-called follow-the-leader (FtL) model on the line. In this simple model, vehicles are on a single line and are not allowed to overtake each other. Another key assumption of the FtL model is that the behavior of a vehicle depends only on the distance to the vehicle in front of it.

Let $N \in \mathbb{N}$ be the number of vehicles and, for $i \in \{1, \dots, N\}$, let $X^i(t)$ the position of vehicle labelled i at time t. As the vehicles cannot overtake each other, we can assume without loss of generality that $X^i(t) \leqslant X^{i+1}(t)$ for any i and t. The motion of the vehicles is defined by the simple ODE

$$\dot{X}^i(t) = V(X^{i+1}(t) - X^i(t)), \qquad t \geqslant 0,\ i \in \{1, \dots, N-1\}.$$

Here we assume that $V : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, Lipschitz continuous and bounded. To avoid collision, one generally assumes the existence of a threshold $e_{\min} > 0$ such that $V \equiv 0$ on $[0, e_{\min}]$. There is an ambiguity on the velocity of the right-most vehicle: it is in general assumed that

$$\dot{X}^N(t) = \max V.$$

One easy checks the following:

Proposition 2.1. *Let V satisfy the conditions above and $(X_0^i)_{i=1,\dots,N}$ be an initial condition satisfying*

$$X_0^i < X_0^{i+1} \qquad \forall i \in \{1, \dots, N-1\}.$$

Then there exists a unique solution to the FtL model. In addition

$$\min_{i=1,\dots,N-1} X^{i+1}(t) - X^i(t) \geqslant \min_{i=1,\dots,N-1} X_0^{i+1} - X_0^i \qquad \forall t \geqslant 0.$$

As N is large (which is a natural assumption) it is however difficult to have a clear idea of the collective behavior of the vehicles. For this reason we discuss next continuous equations describing the density of the vehicles. Then we will present the relationship between FtL and the continuous models.

2.2 A first continuous models: LWR

The LWR model (from Lighthill and Whitham [39] and Richards [46]) is the following conservation law:

$$\partial_t \rho + \partial_x (f(\rho)) = 0 \qquad \text{in } (0, \infty) \times \mathbb{R} \tag{1}$$

Here $f : [0, R] \to \mathbb{R}$ is Lipschitz continuous, nonnegative and such that $f(0) = f(R) = 0$ for some $R > 0$ which stands for the maximal density. One often assumes that f is concave (or semi-concave, i.e., increasing on $[0, a]$ and decreasing on $[a, R]$ for some $a \in (0, R)$).

Let us first explain equation (1). We argue here as if data and solutions are smooth. The idea in this macroscopic model is that each (tiny) vehicle x has a velocity given by a function v of the density. Let $X(t, x)$ be the integral flow of this ODE: $\frac{d}{dt}X(t, x) = v(\rho(t, X(t, x)))$, with $X(0, x) = x$. Then the density $\rho(t)$ is the image by $x \to X(t, x)$ of the initial density ρ_0. Thus, for any smooth test function ϕ with a compact support in $(0, \infty) \times \mathbb{R}$,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} \phi(t,x)\rho(t,x)dx &= \frac{d}{dt}\int_{\mathbb{R}} \phi(t,X(t,x))\rho_0(x)dx = \int_{\mathbb{R}} (\partial_t\phi(t,X(t,x)) + \partial_x\phi(t,X(t,x))\frac{d}{dt}X(t,x))\rho_0(x)dx \\ &= \int_{\mathbb{R}} (\partial_t\phi(t,X(t,x)) + \partial_x\phi(t,X(t,x))v(\rho(t,X(t,x))))\rho_0(x)dx \\ &= \int_{\mathbb{R}} (\partial_t\phi(t,x) + \partial_x\phi(t,x)v(\rho(t,x)))\rho(t,x)dx.\end{aligned}$$

If we set $f(p) = pv(p)$, integrating in time the previous equality leads to

$$0 = \int_0^\infty \frac{d}{dt}\int_{\mathbb{R}} \phi(t,x)\rho(t,x)dx = \int_0^\infty \int_{\mathbb{R}} (\partial_t\phi(t,x)\rho(t,x) + \partial_x\phi(t,x)f(\rho(t,x))dx.$$

Thus ρ is a weak solution to (1).

Let us recall that in general there is no classical solution to (1) and that solutions in the sense of distributions are no unique. To single out a particular solution, one is led to introduce the notion of entropy solution.

Definition 2.2. *A (Kruzhkov) entropy solution to* (1) *is a map $\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ such that $\rho \in [0, R]$ a.e. and, such that, for any nonnegative C^1 function ϕ with compact support in $(0, \infty) \times \mathbb{R}$ and any constant $k \in [0, R]$, one has*

$$\int_0^\infty \int_{\mathbb{R}} |\rho - k|\partial_t\phi + \mathrm{sgn}(\rho - k)(f(\rho) - f(k))\partial_x\phi \geqslant 0. \tag{2}$$

The pair of maps $(\eta, q)(u, k) = (|u - k|, \mathrm{sgn}(u - k)(f(y) - f(k)))$ is an example of entropy pairs, i.e., satisfying $q' = \eta' f'$. Note that the condition reads in a more compact way: for any $k \in [0, R]$,

$$\partial_t(\eta(u, k)) + \partial_x(q(u, k)) \leqslant 0$$

in the sense of distribution. It turns out that under the conditions on f above, the solution ρ depends continuously on time, i.e., $\rho \in C^0([0, \infty), L^1_{loc}(R))$.

Theorem 2.3 (Kruzhkov [38])**.** *Assume that $\rho_0 \in L^\infty(\mathbb{R}, [0, R])$. Then there exists a unique entropy solution to* (1) *with initial condition ρ_0.*

In addition, the equation preserves mass

$$\int_{\mathbb{R}} \rho(t, x)dx = \int_{\mathbb{R}} \rho_0(x)dx \qquad \forall t \geqslant 0$$

and is L^1-contractive: if $\tilde{\rho}$ is another solution of (1) *with initial condition $\tilde{\rho}_0 \in L^\infty(\mathbb{R}, [0, R]) \cap L^1(\mathbb{R})$, then*

$$\int_{\mathbb{R}} |\rho(t, x) - \tilde{\rho}(t, x)|dx \leqslant \int_{\mathbb{R}} |\rho_0(x) - \rho_0(x)|dx \qquad \forall t \geqslant 0.$$

Remark 2.4. We have seen above that the semi-group defined by Equation (1) preserves the mass. Then it is known [25] that it is L^1-contractive if and only if it satisfies a comparison principle: if ρ_0^1, ρ_0^2 are two initial conditions in $L^\infty(\mathbb{R}, [0, R])$, with $\rho_0^1 \leqslant \rho_0^2$ a.e., then the associated solutions ρ^1 and ρ^2 satisfy: $\rho^1 \leqslant \rho^2$ a.e.

For a proof of Theorem 2.3, see for instance [12, 26, 38]. The proof of the uniqueness is based on the entropy formulation and a doubling variable technique consisting in replacing the constant k in the entropy formulation (2) by the other solution $\tilde{\rho}(s,y)$, integrate also in (s,y). Choosing a suitable penalization in the test function gives the uniqueness and the L^1-contraction. Let us underline an intermediate step towards the L^1-contraction, which will be used later (see again [12, 26] for instance).

Proposition 2.5 (Kato's inequality)**.** *If ρ and $\tilde{\rho}$ are two entropy solution to (1) starting from $\rho_0 \in L^\infty(\mathbb{R},[0,R])$ and $\tilde{\rho}_0 \in L^\infty(\mathbb{R},[0,R])$ respectively, then, for any test function $\phi \in C^1_c([0,\infty)\times\mathbb{R})$ and any $t>0$,*

$$\begin{aligned}\int_0^\infty &|(\rho-\tilde{\rho})(t,x)|\phi(t,x)dx \\ &-\int_0^t\int_{\mathbb{R}} |(\rho-\tilde{\rho})(s,x)|\partial_t\phi(s,x) + \mathrm{sgn}(\rho(s,x)-\tilde{\rho}(s,x))(f(\rho(s,x))-f(\tilde{\rho}(s,x)))\partial_x\phi(s,x) \\ &\qquad\qquad \leqslant \int_0^\infty |(\rho_0-\tilde{\rho}_0)(x)|\phi(0,x)dx.\end{aligned}$$

Taking $\phi \equiv 1$ gives the L^1-contraction. In fact, one can derive from this inequality a finite speed of propagation property.

Let us finally underline that there exists many CL models in traffic flow. We only discuss here the simplest one. Another very popular one is a so-called second-order model (also called ARZ model, after Aw-Rascle [4] and Zhang [49]), which involves an additional equation for the pressure: Namely,

$$\begin{cases} \partial_t\rho + \partial_x(\rho v) = 0 \\ \partial_t[v+p(\rho)] + \partial_x[\rho v(v+p(\rho))] = 0 \end{cases}$$

The unknown are the density ρ and velocity v. The other conserved variable $[\rho(v+p(\rho))]$ is the so-called generalized momentum of the system. The given function p is the pressure function and accounts for drivers' reactions to the state of traffic in front of them. The paper [5] explains that the particle counterpart of this system takes the form

$$\begin{cases} \dot{x}^i(t) = V^i \\ \dot{V}^i = p'(\frac{1}{x^{i+1}-x^i})\frac{V^{i+1}-V^i}{(x^{i+1}-x^i)^2} \end{cases}$$

See also [28] for a micro-macro derivation. Let us finally note that another popular second order discrete model—the so-called Bando model [6]—leads to the more classical LWR model after scaling [32].

2.3 A second continuous models: the Hamilton-Jacobi equation

A second equation naturally associated with the traffic flow is the Hamilton-Jacobi equation:

$$\partial_t u(t,x) + f(\partial_x u(t,x)) = 0 \tag{3}$$

The Hamiltonian $f:[0,R]\to\mathbb{R}$ is the same as above: it is Lipschitz continuous, nonnegative and such that $f(0)=f(R)=0$ for some $R>0$. This Hamilton-Jacobi (HJ) equation is a kind of integrated version of (1). In general the solution of (3) is expected to be at most Lipschitz continuous (no classical solution). The correct notion of solution is the notion of viscosity solution.

Definition 2.6. *A Lipschitz continuous map $u:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is a viscosity subsolution (resp. supersolution) to (3) if, for any C^1 test function $\phi=\phi(t,x)$ such that $u-\phi$ has a strict local maximum (resp. strict local minimum) at some point $(\bar{t},\bar{x})\in(0,\infty)\times\mathbb{R}$, one has*

$$\partial_t\phi(\bar{t},\bar{x}) + f(\partial_x\phi(\bar{t},\bar{x})) \leqslant 0 \qquad (resp. \geqslant 0).$$

A viscosity solution is at the same time a subsolution and a supersolution.

One easily checks that (in our context of Lipschitz continuous solution), a viscosity solution satisfies the equation a.e.. The converse is false in general.

Theorem 2.7. *Let $u_0 : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous and such that $u_0' \in [0, R]$ a.e. Then there exists a unique viscosity solution u to* (3) *satisfying the initial condition $u(0, \cdot) = u_0$.*

In addition, the equation preserves L^∞−bounds and is L^∞−contractive: if u_0^1, u_0^2 are two initial conditions such that $(u_0^1)', (u_0^2)' \in [0, R]$ a.e, then the associated solutions u^1 and u^2 satisfy

$$\inf u_0^1 \leqslant u^1 \leqslant \max u_0^1 \qquad \forall (t, x) \in [0, \infty) \times \mathbb{R}$$

and

$$\|u^1 - u^2\|_\infty \leqslant \|u_0^1(x) - u_0^2\|_\infty.$$

One can also show that the solution is stable with respect to locally uniform perturbation of the Hamiltonian or of the initial condition. We refer to the classical references [7, 8, 24, 40]. The proof of the uniqueness and the L^∞−contraction is an easy consequence of the following comparison principle (which has—surprisingly—nothing to do with the comparison principle discussed for the CL):

Theorem 2.8 (Comparison)**.** *Let u and v be respectively a subsolution and a supersolution to* (3)*. If $u(0, \cdot) \leqslant v(0, \cdot)$ in $\mathbb{R}$, then $u \leqslant v$ in $[0, \infty) \times \mathbb{R}$.*

There is actually a strong relationship between the scalar conservation law (1) and the Hamilton-Jacobi equation (3). Namely:

Proposition 2.9. *If u is a viscosity solution to* (3) *in an open set $\mathcal{O} \subset (0, \infty) \times \mathbb{R}$, then $\rho := \partial_x u$ is an entropy solution to* (1) *in $\mathcal{O}$.*

Sketch of proof. We explain the argument when f is smooth, $\mathcal{O} = (0, \infty) \times \mathbb{R}$ and u is the solution to (3) with a smooth initial condition u_0 while ρ solves (1) with initial condition $\rho_0 = \partial_x u_0$. The case of a general case is explained in Appendix A.3 of [19].

The idea is that the entropy solution of the scalar conservation law (1) as well as the viscosity solution of the HJ equation (3) can both be obtained as a vanishing viscosity limit: $\rho = \lim_{\epsilon \to 0^+} \rho^\epsilon$ (in L^1_{loc}) and $u = \lim_{\epsilon \to 0^+} u^\epsilon$ (in L^∞_{loc}) where ρ^ϵ and u^ϵ are the unique classical solution to

$$\partial_t \rho^\epsilon + \partial_x (f(\rho^\epsilon)) = \epsilon \partial_{xx} \rho^\epsilon, \qquad \rho^\epsilon(0, \cdot) = \rho_0$$

and

$$\partial_t u^\epsilon(t, x) + f(\partial_x u^\epsilon(t, x)) = \epsilon \partial_{xx} u^\epsilon(t, x), \qquad u^\epsilon(0, \cdot) = u_0.$$

On the other hand, in this smooth case, it is clear that, $\partial_x u^\epsilon$ is a smooth solution to the viscous conservation law. By uniqueness, $\rho^\epsilon = \partial_x u^\epsilon$. Letting $\epsilon \to 0$ gives the equality $\rho = \partial_x u$. □

2.4 A derivation of the LWR model from the follow-the-leader model

One of the interests to link between the scalar conservation law and the HJ equation is that it allows a simple justification of the passage from the Follow-the-Leader model to the conservation law.

For $N > 0$ large, let $(X_0^i)_{i=1,\ldots,N}$ be an initial condition for the FtL model satisfying (to fix the ideas)

$$X_0^i + e_{\min} \leqslant X_0^{i+1} \qquad \forall i \in \{1, \ldots, N-1\}. \tag{4}$$

(recall that $e_{\min} > 0$ is such that $V = 0$ on $[0, e_{\min}]$). Let (X^i) be the associated solution to the FtL model:

$$\begin{cases} \dot{X}^i(t) = V(X^{i+1}(t) - X^i(t)) \qquad \forall i \in \{1, \ldots, N-1\}, \qquad \dot{X}^N(t) = \max V, \qquad t \geqslant 0, \\ X^i(0) = X_0^i \qquad \forall i \in \{1, \ldots, N\}. \end{cases} \tag{5}$$

We define the (scaled) empirical measures

$$\rho^N(t) = \frac{1}{N} \sum_{i \in \{0, \ldots, N\}} \delta_{N^{-1} X^i(Nt)}, \qquad t \geqslant 0, \tag{6}$$

where δ_a is the Dirac mass at $a \in \mathbb{R}$. The ratio $1/N$ in front of the sum is clear: we thus obtain a probability on $\mathbb{R}$. The whole point is that we want to see a continuous density, and thus we have to scale space by N^{-1}. The scaling in time by N is here to see some motion to the scaled flow.

Theorem 2.10 (Micro-Macro derivation on the line)**.** *Assume that the initial condition*

$$\rho_0^N = \frac{1}{N} \sum_{i \in \{0,\dots,N\}} \delta_{N^{-1}X_0^i} \tag{7}$$

satisfies (4) *and converges weakly-* in the sense of measures to some density* ρ_0 *as* $N \to \infty$*. Then* $\rho_0 \in [0, R]$ *a.e. and* (ρ^N) *converges in the sense of distributions and as* $N \to \infty$ *to the entropy solution* ρ *of the LWR model with* $f(p) := pV(1/p)$ *and with initial condition* ρ_0*.*

Many derivation of this result—with variants—are known: see [5, 22, 27, 34, 35] to quote only a few. The argument developed below are relatively elementary: the proof relies on simple estimates and on the relationship between CL and HJ equation as explained in Proposition 2.9. In fact, as proved in [27], one can show that (under stronger assumptions on the flux and with much more work), that a regularized version of the empirical density is "almost" an entropy solution to the scalar conservation law; in fact it actually enjoys BV estimates which are typical of scalar conservation laws: see below.

Note that, as $V : \mathbb{R}_+ \to \mathbb{R}_+$ is Lipschitz continuous, nondecreasing and bounded, and satisfies $V = 0$ on $[0, e_{\min}]$ for some $e_{\min} > 0$, then $f : [0, R] \to \mathbb{R}_+$ given by $f(p) = pV(1/p)$ is Lipschitz continuous in $(0, \infty)$ and satisfies $f(0) = f(R) = 0$ for $R = 1/e_{\min}$. If in addition V is concave in $[e_{\min}, \infty)$, then f is concave and Lipschitz in $[0, R]$.

A typical example of map V is given by $V(e) = A - Bx^{-1}$ is $e > e_{\min} := A^{-1}B$ (for some $A, B > 0$). Then $f(r) = Ar - Br^2$ for $r \in [0, R]$, with $R := AB^{-1}$.

Sketch of proof of Theorem 2.10. Let $u^N(t, x) = \rho^N(t, (-\infty, x])$ for any t. Note that $\partial_x u^N = \rho^N$ in the sense of distributions. Moreover the u^N are nondecreasing in space and nonincreasing in time (because the X^i are increasing). Finally, u^N is u.s.c. The main part of the proof consists in checking that u^N converges to the unique viscosity solution u to the HJ equation (3) with initial condition u_0, where $u_0(x)$ is the antiderivative of ρ_0 which vanishes at $-\infty$.

Let us first assume the convergence for a while and explain how to conclude. As $\partial_x u^N = \rho^N$ is the sense of distributions, and as u^N converges to u, ρ^N converges to $\rho := \partial_x u$ in the sense of distribution. But we know by Proposition 2.9 that, as u solves the HJ equation, $\rho = \partial_x u$ is the entropy solution ρ of the LWR model with $f(p) := pV(1/p)$ and with initial condition ρ_0. This shows the convergence of ρ^N to ρ.

We now turn to the proof of the convergence of u^N. We first check that the sequence u^N is relatively compact for the local uniform convergence and that any cluster point is Lipschitz continuous, with $\partial_x u \in [0, R]$. Indeed, by Proposition 2.1, the X^i satisfy

$$\min_{i=1,\dots,N-1} X^{i+1}(t) - X^i(t) \geqslant \min_{i=1,\dots,N-1} X_0^{i+1} - X_0^i \geqslant e_{\min} \qquad \forall t \geqslant 0.$$

Thus, on a (microscopic) space interval $[A, B]$, there are at most $(B - A)/e_{\min} + 1$ vehicles. Hence, at the microscopic scaling, if $a \leqslant b$,

$$0 \leqslant u^N(t, b) - u^N(t, a) = \rho^N(t, (a, b]) \leqslant (b - a)/e_{\min} + 1/N. \tag{8}$$

This shows that the u^N are almost Lipschitz in space. On the other hand, if we fix a vehicle $i \in \{1, \dots, N\}$, then $t \to u^N(t, N^{-1}X^i(Nt))$ is constant (and equal to $N^{-1}(i - 1)$). As the X^i are uniformly Lipschitz, we get, for $0 \leqslant s \leqslant t$,

$$\begin{aligned} u^N(s, N^{-1}X^i(Ns)) = u^N(t, N^{-1}X^i(Nt)) &\leqslant u^N(t, N^{-1}X^i(Ns) + C(t - s)) \\ &\leqslant u^N(t, N^{-1}X^i(Ns)) + C(t - s) + 1/N. \end{aligned}$$

As u^N is nonincreasing in time, we infer that, for $x = N^{-1}X^i(Ns)$,

$$|u^N(s, x) - u^N(t, x)| \leqslant C(t - s) + 2/N.$$

As u^N is constant between two $N^{-1}X^i(Ns)$ we can conclude that the u^N are almost uniformly Lipschitz continuous in time as well. Finally we note that $u^N \in [0,1]$ for any N. Thus the sequence u^N is relatively compact for the local uniform convergence. In addition if u is any cluster point, then estimate (8) implies that $0 \leqslant u(t,b)-u(t,a) \leqslant (b-a)/e_{\min}$ if $a < b$, so that $\partial_x u \in [0,R]$ a.e., since $R = 1/e_{\min}$.

We now assume that (u^N) converges (along a subsequence denoted in the same way) to a map u locally uniformly. We claim that u is the unique viscosity solution to (3) with initial condition u_0. This is enough to show that the whole sequence (u^N) converges to u. We note that u is Lipschitz continuous and satisfies the initial condition, as well as the condition $\partial_x u \in [0,R]$.

We will only explain the subsolution property, as the supersolution is similar (but slightly more technical). Let ϕ be a C^1 test function such that $u-\phi$ has a strict local maximum at some point $(\bar{t},\bar{y}) \in (0,\infty)\times\mathbb{R}$. We have to check that

$$\partial_t\phi(\bar{t},\bar{y}) + f(\partial_x\phi(\bar{t},\bar{y})) \leqslant 0. \tag{9}$$

Note that, if $\partial_x\phi(\bar{t},\bar{y}) = 0$, then the result is obvious because, as u is nonincreasing in time, $\partial_t\phi(\bar{t},\bar{y}) \leqslant 0$ while $f(\partial_x\phi(\bar{t},\bar{y})) = f(0) = 0$. From now on we assume that $\partial_x\phi(\bar{t},\bar{y}) > 0$ (recall that u is nondecreasing in space, so that necessarily $\partial_x\phi(\bar{t},\bar{y}) \geqslant 0$).

As u^N converges to u locally uniformly and $u-\phi$ has a strict local maximum at $(\bar{t},\bar{y})$ (recall also that u^N is u.s.c.), standard argument in viscosity solutions [7, 8, 24, 40] show the existence of $(t^N,y^N) \to (\bar{t},\bar{y})$ such that $u^N-\phi$ has a local maximum at (t^N,y^N). This means that, for (t,x) close to (t^N,y^N),

$$u^N(t,x) \leqslant \phi(t,x) + u^N(t^N,y^N) - \phi(t^N,y^N). \tag{10}$$

We note that, if N is large enough, there exists $i^N \in \{1,\dots,N-1\}$ such that $y^N = N^{-1}X^{i^N}(t^N)$. Indeed, otherwise u^N is locally constant near (t^N,y^N), and (10) contradicts the fact that $\partial_x\phi(t^N,y^N) > 0$ for N large enough.

Recall now that $t \to u^N(t,N^{-1}X^{i^N}(Nt))$ is constant in time. Choosing $x = N^{-1}X^{i^N}(Nt)$ in (10) and taking the derivative in time of the resulting expression, we infer that

$$\begin{aligned} 0 &= \partial_t\phi(t^N,y^N) + \partial_x\phi(t^N,y^N)\dot{X}^{i^N}(Nt^N) \\ &= \partial_t\phi(t^N,y^N) + \partial_x\phi(t^N,y^N)V(X^{i^N+1}(Nt^N) - X^{i^N}(Nt^N)), \end{aligned} \tag{11}$$

where we used (5) in the second equality. Note that, if $(X^{i^N+1}(Nt^N) - X^{i^N}(Nt^N))$ tends to infinity along a subsequence, then passing to the limit in (11) along this subsequence yields to

$$0 = \partial_t\phi(\bar{t},\bar{x}) + \partial_x\phi(\bar{t},\bar{y})\max V \geqslant \partial_t\phi(\bar{t},\bar{y}) + f(\partial_x\bar{\phi}(t,x)).$$

Thus (9) holds.

We now assume that $(X^{i^N+1}(Nt^N) - X^{i^N}(Nt^N))$ is bounded. By the very definition of u^N and ρ^N, $u^N(t^N,N^{-1}X^{i^N+1}(Nt^N)) = u^N(t^N,y^N) + 1/N$. Hence, still by (10),

$$\begin{aligned} u^N(t^N,X^{i^N+1}(Nt^N)) &= u^N(t^N,y^N) + 1/N \\ &\leqslant \phi(t^N,N^{-1}X^{i^N+1}(t^N)) + u^N(t^N,y^N) - \phi(t^N,N^{-1}X^{i^N}(t^N)). \end{aligned}$$

Hence

$$\begin{aligned} 1/N &\leqslant \phi(t^N,N^{-1}X^{i^N+1}(Nt^N)) - \phi(t^N,N^{-1}X^{i^N}(Nt^N)) \\ &= \partial_x\phi(t^N,y^N)N^{-1}(X^{i^N+1}(t^N) - X^{i^N}(t^N)) + o(1/N). \end{aligned}$$

Recalling that $\partial_x\phi(t^N,y^N) > 0$, we infer that

$$X^{i^N+1}(t^N) - X^{i^N}(t^N) \geqslant \frac{1}{\partial_x\phi(t^N,y^N)} + o(1).$$

As V is nondecreasing and Lipschitz, we get from (11) and the definition of f that

$$0 \geqslant \partial_t \phi(t^N, y^N) + \partial_x \phi(t^N, y^N) V(1/\partial_x \phi(t^N, y^N)) - o(1) = \partial_t \phi(t^N, y^N) + f(\partial_x \phi(t^N, y^N)) - o(1).$$

We obtain inequality (9) by letting $N \to \infty$. $\square$

2.5 Extensions

2.5.1 Stronger convergence results

In fact the convergence in Theorem 2.10 can be strongly sharpened. One of the very nice discoveries of [27] is to explain that the solution of the follow-the-leader model almost solves the LWR equation. To this see, let (X^i) be a solution to the FtL model (5) and let us set

$$y^i(t) = \frac{1}{X^{i+1}(Nt) - X^i(Nt)}, \qquad x^i(t) = N^{-1} X^i(Nt).$$

We introduce the discrete density

$$\tilde{\rho}^N(t,x) = \sum_{i=1}^{N-1} y^i(t) \mathbf{1}_{[x^i(t), x^{i+1}(t))}.$$

To obtain a convergence of $\tilde{\rho}^N$, we need to suppose stronger conditions on the flux.

Assumption on the flux. we assume that the flux $f : [0,R] \to \mathbb{R}_+$ is of class C^2 and uniformly concave, with $f(0) = f(R) = 0$. We denote by $V : \mathbb{R}_+ \to \mathbb{R}_+$ the velocity of the discrete model. It is defined by $V(e) = ef(1/e)$ for $e \in [e_{\min}, \infty)$, where $e_{\min} := 1/R$, and $V(e) = 0$ on $[0, e_{\min}]$. We note that V is Lipschitz continuous, nondecreasing and bounded on $\mathbb{R}_+$, and of class C^2 on $(e_{\min}, \infty)$. We set $V_{\max} = \max V = f'(0)$. Finally, setting $v(r) = V(1/r)$, we assume that $r \to rv'(r)$ is nonincreasing on $(0, \rho_{\max})$.

Example. For instance, if $f(r) = Ar - Br^2$ (for $A, B > 0$) and $\rho_{\max} = AB^{-1}$, then $v(r) = A - Br$ is defined on $[0, AB^{-1}]$ and is such that $r \to rv'(r) = -Br$ is nonincreasing, while V, defined by $V(x) = 0$ on $[0, e_{\min}]$ with $e_{\min} = A^{-1}B = 1/\rho_{\max}$ and $V(x) = A - Bx^{-1}$ if $x > e_{\min}$, is nondecreasing and bounded.

Theorem 2.11 (Sharp estimates on $\tilde{\rho}^N$ [27])**.** *Under our assumptions above, we have:*

- *(BV estimate) There is a constant C, depending on the diameter of the support of $\tilde{\rho}^N(0,\cdot)$, such that, for any $t \geqslant 0$,*

$$TV(v(\tilde{\rho}^N(t,\cdot))) \leqslant \min\{TV(v(\tilde{\rho}^N(0,\cdot))), C(1+t^{-1})\}.$$

- *(Approximate entropy inequality) For any $k \geqslant 0$ and any $\phi \in C_c^1((0,\infty)\times\mathbb{R})$ with $\phi \geqslant 0$,*

$$\int_0^\infty \int_{\mathbb{R}} |\tilde{\rho}^N - k|\phi_t + \mathrm{sgn}(\tilde{\rho}^N - k)(f(\tilde{\rho}^N) - f(k))\phi_x \geqslant -N^{-1} \sup_t TV(v(\tilde{\rho}^N(t,\cdot))) \int_0^\infty \|\phi_x(t,\cdot)\|_\infty dt. \quad (12)$$

Note that (12) means that $\tilde{\rho}^N$ is an approximate entropy solution of the conservation law (1). Using additional bounds on $\tilde{\rho}^N$ allow to pass to the limit in (12) and prove the L^1 convergence of $\tilde{\rho}^N$ to the unique solution $\tilde{\rho}$ of (1): see Theorem 2.3. in [27].

Proof. We check below the proof of the inequality

$$TV(v(\tilde{\rho}^N(t,\cdot))) \leqslant TV(v(\tilde{\rho}^N(0,\cdot))). \quad (13)$$

The proof of the very nice inequality

$$TV(v(\tilde{\rho}^N(t,\cdot))) \leqslant C(1+t^{-1})$$

is more technical and given in [27, Proposition 3.6]. To prove (13) we first note that

$$TV(v(\tilde{\rho}^N(t,\cdot))) = \sum_{i=0}^{N-1} \left| v(y^{i+1}(t)) - v(y^i(t)) \right|,$$

where, for $i = 1, \dots, N-1$,

$$y^i(t) := \frac{1}{X^{i+1}(Nt) - X^i(Nt)}$$

and where, by convention, $X^0(t) = -\infty$, $X^{N+1}(t) = \infty$ and thus $y^0(t) = y^N(t) = 0$.

We note that, for $i = 1, \dots, N-1$,

$$\dot{y}^i(t) = -(y^i(t))^2 N \left(v(y^{i+1}(t)) - v(y^i(t)) \right).$$

Thus

$$\begin{aligned}
\frac{d}{dt} TV(v(\tilde{\rho}^N(t,\cdot))) &= \sum_{i=0}^{N-1} \operatorname{sgn}\left(v(y^{i+1}(t)) - v(y^i(t)) \right) \left(v'(y^{i+1}(t))\dot{y}^{i+1}(t) - v'(y^i(t))\dot{y}^i(t) \right) \\
&= -\operatorname{sgn}\left(v(y^1(t)) - v(y^0(t)) \right) v'(y^0(t))\dot{y}^0(t) + \operatorname{sgn}\left(v(y^N(t)) - v(y^{N-1}(t)) \right) v'(y^N(t))\dot{y}^N(t) \\
&\quad + \sum_{i=1}^{N-1} v'(y^i(t))\dot{y}^i(t) \left(\operatorname{sgn}\left(v(y^i(t)) - v(y^{i-1}(t)) \right) - \operatorname{sgn}\left(v(y^{i+1}(t)) - v(y^i(t)) \right) \right).
\end{aligned}$$

The first two terms vanish. As, for $i = 1, \dots, N-1$,

$$\begin{aligned}
&v'(y^i(t))\dot{y}^i(t) \left(\operatorname{sgn}\left(y^i(t) - y^{i-1}(t) \right) - \operatorname{sgn}\left(y^{i+1}(t) - y^i(t) \right) \right) \\
&= -v'(y^i(t))(y^i(t))^2 N \left(v(y^{i+1}(t)) - v(y^i(t)) \right) \left(\operatorname{sgn}\left(v(y^i(t)) - v(y^{i-1}(t)) \right) - \operatorname{sgn}\left(v(y^{i+1}(t)) - v(y^i(t)) \right) \right) \leqslant 0,
\end{aligned}$$

since $z \to v(z)$ is non increasing, we infer that

$$\frac{d}{dt} TV(\tilde{\rho}^N(t,\cdot))) \leqslant 0.$$

We now check that $\tilde{\rho}^N$ is an approximate entropy solution, which is actually the key argument of the proof of [27, Theorem 2.3]. Fix $k \geqslant 0$ and $\phi \in C_c^1$ with $\phi \geqslant 0$. We set

$$x^i(t) = N^{-1} X^i(Nt) \text{ for } i = 1, \dots, N, \ x^0(t) = -\infty, \ x^{N+1}(t) = \infty,$$

and compute

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}} \left| \tilde{\rho}^N(t) - k \right| \phi(t,x)dx = \frac{d}{dt} \sum_{i=0}^{N} \int_{x^i(t)}^{x^{i+1}(t)} \left| y^i(t) - k \right| \phi(t,x)dx \\
&= \int_{\mathbb{R}} \left(\left| \tilde{\rho}^N(t,x) - k \right| \phi_t(t,x) + \operatorname{sgn}(\tilde{\rho}^N(t,x) - k)(f(\tilde{\rho}^N(t,x)) - f(k))\phi_x(t,x) \right) dx + R(t)
\end{aligned}$$

where

$$\begin{aligned}
R(t) &= \sum_{i=0}^{N} \int_{x^i(t)}^{x^{i+1}(t)} \operatorname{sgn}\left(y^i(t) - k \right) \left[\dot{y}^i(t)\phi(t,x) - (y^i(t)v(y^i(t)) - kv(k))\phi_x(t,x) \right] dx \\
&\qquad + \sum_{i=0}^{N} \left| y^i(t) - k \right| \left[\phi(t,x^{i+1}(t))\dot{x}^{i+1}(t) - \phi(t,x^i(t))\dot{x}^i(t) \right] \\
&= \sum_{i=1}^{N-1} J^i(t) + \sum_{i=1}^{N} K^i(t)\phi(t,x^i(t)),
\end{aligned}$$

with, for $i = 1, \dots, N-1$,

$$\begin{aligned} J^i(t) &= \operatorname{sgn}\left(y^i(t) - k\right) \frac{\dot{y}^i(t)}{N y^i(t)} \left(N y^i(t) \int_{x^i(t)}^{x^{i+1}(t)} \phi(t,x) dx - \phi(t, x^{i+1}(t)) \right) \\ &= -\, y^i(t) \operatorname{sgn}\left(y^i(t) - k\right) \left(v(y^{i+1}(t)) - v(y^i(t))\right) \left(N y^i(t) \int_{x^i(t)}^{x^{i+1}(t)} (\phi(t,x) - \phi(t, x^{i+1}(t))) dx \right) \end{aligned}$$

and, for $i = 2, \dots, N-1$,

$$\begin{aligned} K^i(t) = &\operatorname{sgn}\left(y^{i-1}(t) - k\right) \frac{\dot{y}^{i-1}(t)}{N y^{i-1}(t)} \\ &+ \operatorname{sgn}\left(y^i(t) - k\right) \left(y^i(t) v(y^i(t)) - k v(k)\right) - \operatorname{sgn}\left(y^{i-1}(t) - k\right) \left(y^{i-1}(t) v(y^{i-1}(t)) - k v(k)\right) \\ &+ \dot{x}^i(t) \left(\left|y^{i-1}(t) - k\right| - \left|y^i(t) - k\right| \right) \\ = &-\, y^{i-1}(t) \operatorname{sgn}\left(y^{i-1}(t) - k\right) \left(v(y^i(t)) - v(y^{i-1}(t))\right) \\ &+ \operatorname{sgn}\left(y^i(t) - k\right) \left(y^i(t) v(y^i(t)) - k v(k)\right) - \operatorname{sgn}\left(y^{i-1}(t) - k\right) \left(y^{i-1}(t) v(y^{i-1}(t)) - k v(k)\right) \\ &+ v(y^i(t)) \left(\left|y^{i-1}(t) - k\right| - \left|y^i(t) - k\right| \right) \\ = &\, k(v(y^i(t)) - v(k)) \left(\operatorname{sgn}\left(y^i(t) - k\right) - \operatorname{sgn}\left(y^{i-1}(t) - k\right) \right) \leqslant 0, \end{aligned}$$

since v is nonincreasing. Finally

$$\begin{aligned} K^1(t) &= -k v(k) + \operatorname{sgn}(y^1 - k) \left(y^1 v(y^1) - k v(k)\right) + k v(y^1) - |y^1 - k| v(y^1) \\ &= k \left(v(y^1) - v(k)\right) \left(1 + \operatorname{sgn}(y^1 - k)\right) \leqslant 0 \end{aligned}$$

and

$$\begin{aligned} K^N(t) = &\operatorname{sgn}\left(y^{N-1}(t) - k\right) \frac{\dot{y}^{N-1}(t)}{N y^{N-1}(t)} + \operatorname{sgn}(y^N - k) \left(y^N v(y^N) - k v(k)\right) \\ &- \operatorname{sgn}(y^{N-1} - k) \left(y^{N-1} v(y^{N-1}) - k v(k)\right) + \dot{y}^N (|y^{N-1} - k| - |y^N - k|) \\ = &-\, y^{N-1} \operatorname{sgn}(y^{N-1} - k) \left(v(y^N) - v(y^{N-1})\right) + \operatorname{sgn}(y^N - k) \left(y^N v(y^N) - k v(k)\right) \\ &- \operatorname{sgn}(y^{N-1} - k) \left(y^{N-1} v(y^{N-1}) - k v(k)\right) + v(y^N)(|y^{N-1} - k| - |y^N - k|) \\ = &k \left(v(k) - V_{\max}\right) (1 + \operatorname{sgn}(y^{N-1} - k)) \leqslant 0. \end{aligned}$$

We note that

$$J^i(t) \leqslant |y^i(t)| \left|v(y^{i+1}(t)) - v(y^i(t))\right| \|\phi_x(t,\cdot)\|_\infty |x^{i+1}(t) - x^i(t)|,$$

so that

$$\sum_{i=1}^{N-1} J^i(t) \leqslant N^{-1} TV(v(\tilde{\rho}^N(t)) \|\phi_x(t,\cdot)\|_\infty.$$

As

$$0 = \int_0^\infty \int_{\mathbb{R}} \left(\left|\tilde{\rho}^N(t,x) - k\right| \phi_t(t,x) + \operatorname{sgn}(\tilde{\rho}^N(t,x) - k)(f(\tilde{\rho}^N(t,x)) - f(k)) \phi_x(t,x) \right) dx dt + \int_0^\infty R(t) dt,$$

we obtain

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} \left(\left|\tilde{\rho}^N(t,x) - k\right| \phi_t(t,x) + \operatorname{sgn}(\tilde{\rho}^N(t,x) - k)(f(\tilde{\rho}^N(t,x)) - f(k)) \phi_x(t,x) \right) dx dt \\ &\qquad \geqslant -\int_0^\infty \sum_{i=1}^{N-1} J^i(t) dt \geqslant -N^{-1} \sup_t TV(v(\tilde{\rho}^N(t)) \int_0^\infty \|\phi_x(t,\cdot)\|_\infty dt. \end{aligned}$$

□

2.5.2 Inhomogeneous models

A natural extension of Theorem 2.10 concerns the case where vehicles are inhomogeneous (see [21]). Let us describe this setting which will be useful below to handle problems with a bifurcation. To fix the ideas, we explain here a case in which there are trucks and cars on the road. Note that the velocity rule V in the FtL model should certainly depend on the fact that the vehicle is a car or a truck (at the very least!). In order to describe the fact that there is a given proportion p of cars—and thus $(1-p)$ of trucks—for some $p \in (0,1)$ on the road, one is led to introduced a random FtL model:

$$\dot{X}^i(t) = V_{Z^i}(X^{i+1}(t) - X^i(t)),\ \forall i \in \{1, \ldots, N-1\}, \quad \dot{X}^N(t) = \max V_{Z^N}, \qquad \forall t \geqslant 0,$$

where now $V = V_z(x)$ also depends on the type $z \in \{0,1\}$ of the vehicle: say, $Z^i = 1$ if the vehicle is a car and $Z^i = 0$ if it is a truck. We assume that the (Z_i) are independent random variables, with a Bernoulli law of with a parameter p: $\mathbb{P}[Z^i = 1] = p$, $\mathbb{P}[Z^i = 0] = 1-p$. This assumption is natural in the case where vehicles cannot overtake each other, as one can think that then the vehicle have arrived on the road at random. The assumptions on V_z are as before: for $z \in \{0,1\}$, V_z is Lipschitz continuous, increasing on $[v_{z,\min}, \infty)$ and vanish on $[0, v_{z,\min}]$.

We define the measure ρ^N by (6) and note that, in our setting, ρ^N is random.

Theorem 2.12 (Micro-Macro derivation in the inhomogeneous case [21])**.** *Assume that the initial condition ρ_0^N defined in (7) satisfies (4) and converges weakly-* in the sense of measures to some density ρ_0 as $N \to \infty$. Then (ρ^N) converges a.s., in the sense of distributions and as $N \to \infty$, to the entropy solution ρ of the LWR model with initial condition ρ_0 and with a flux $\bar{f}$ defined as follows:*

$$\bar{f}(p) = p\bar{V}(1/p), \tag{14}$$

where, if we set $\bar{v} = \min\{\sup V_0, \sup V_1\}$, then $\bar{V} : [0,\infty) \to [0, \bar{v}]$ is given by

$$\bar{V}(e) = v \iff e = pV_1^{-1}(v) + (1-p)V_0^{-1}(v) \qquad \forall v \in (0, \bar{v}).$$

Remarks 2.13.

1. By the definition of $\bar{V}$, $\bar{V}$ vanishes in $[0, pv_{1,\min} + (1-p)v_{0,\min}]$. On the other hand the maximal speed in the limit problem is the minimum $\bar{v} = \min\{\sup V_0, \sup V_1\}$ of the maximal speeds of each type of vehicle.
2. Let us point out that if one insists to remain determinist and keep track of the type of the vehicles (i.e., the density of cars and the density of trucks), the limit should be a system of conservation laws, somehow related to the ARZ model.
3. We do not know if stronger estimates (as in Theorem 2.11) or stronger convergence results (L^1-convergence for instance) hold in this context.

The proof of Theorem 2.12 is not difficult, but to give all the details would take some place. We only explain the definition of $\bar{f}$. In principle this is a problem of stochastic homogenization of Hamilton-Jacobi equations (see the pioneering work [47]). It turns out that, in contrast with most of these problems, there exists "correctors" in our setting, i.e., self-similar solutions for the discrete model. These self-similar solutions, which we describe now, explain the definition of $\bar{f}$. Let us set $\bar{v} = \min\{\sup V_0, \sup V_1\}$ and fix $v \in (0, \bar{v})$. As the $V_z(\cdot)$ are increasing and continuous, there exists a unique $e_z^v = V_z^{-1}(v)$ such that $V_z(e_z^v) = v$, for $z \in \{0,1\}$. Let us define $(Y_0^{v,i})_{i\in\mathbb{N}}$ by $Y_0^{v,0} = 0$ and $Y_0^{v,i+1} - Y_0^{v,i} = e_{Z_i}^v$ for any $i \in \mathbb{N}$. Then the family $(Y^{v,i})$ defined by

$$Y^{v,i}(t) = Y_0^{v,i} + vt \qquad \forall i \in \mathbb{N},\ t \geqslant 0$$

solves

$$\dot{Y}^{v,i}(t) = v = V_{Z_i}(e_{Z_i}^v) = V_{Z^i}(Y^{v,i+1}(t) - Y^{v,i}(t)), \qquad \forall i \in \mathbb{N}, \quad \forall t \geqslant 0,$$

Therefore $(Y^{v,i})$ is a self-similar solution to the equation. On the other hand,

$$\lim_{i\to\infty} \frac{Y_0^{v,i}}{i} = \lim_{i\to\infty} \frac{1}{i} \sum_{j=0}^{i-1} (Y_0^{v,j+1} - Y_0^{v,j}) = \lim_{i\to\infty} \frac{1}{i} \sum_{j=0}^{i-1} e_{Z_i}^v = \mathbb{E}\left[e_{Z_0}\right] = pe_1^v + (1-p)e_0^v \text{ a.s.}$$

by the law of large numbers. This easily implies that the empirical density

$$\rho_Y^N(t) = N^{-1} \sum_{i \in \mathbb{N}} \delta_{N^{-1} Y^{v,i}(Nt)},$$

weakly converges to the distribution

$$\rho(t, dx) = (pe_1^v + (1-p)e_0^v)^{-1} \mathbf{1}_{[vt,\infty)} dx := \bar{\rho} \mathbf{1}_{[vt,\infty)},$$

where $\bar{\rho} := (pe_1^v + (1-p)e_0^v)^{-1}$. Note that, if Theorem 2.12 holds, then ρ should be at least a weak solution to the CL (1), with $\bar{f}$ instead of f (i.e., a solution to the so-called the Riemann problem with initial condition $\rho_0(x) = (pe_1^v + (1-p)e_0^v)^{-1} \mathbf{1}_{x \geqslant 0})$. This means that, for any test function ϕ with compact support in $(0, \infty) \times \mathbb{R}$,

$$\begin{aligned} 0 &= \int_0^\infty \int_{\mathbb{R}} (\rho \partial_t \phi + f(\rho) \partial_x \phi) dx dt = \int_0^\infty \int_{vt}^\infty (\bar{\rho} \partial_t \phi + f(\bar{\rho}) \partial_x \phi) dx dt \\ &= \int_0^\infty \int_{-\infty}^{x/v} \bar{\rho} \partial_t \phi dt dx - f(\bar{\rho}) \int_0^\infty \phi(t, vt) dt = \bar{\rho} \int_0^\infty \phi(x/v, x) dx - f(\bar{\rho}) \int_0^\infty \phi(t, vt) dt \\ &= (\bar{\rho} v - f(\bar{\rho})) \int_0^\infty \phi(t, vt) dt. \end{aligned}$$

This implies the classical Rankine-Hugoniot condition: $\bar{f}(\bar{\rho}) = \bar{\rho} v$, which is exactly (14).

3 Traffic flow on a 1:1 junction

In this section, we consider an elementary junction, seen at the micro level as a short section of road connecting two long homogeneous (half-)roads. We seek to model for instance a passage from a large road to a smaller one; or to describe a speed bump on a long homogeneous road (or both). When we go from the micro model to the macro one after scaling, the junction reduces to a point. Of course we expect to obtain at the macro model an LWR model on either side of this junction point. On the other hand, the conditions to be put at the junction point itself are less clear. One feels that these conditions must in one way or another reflect the microscopic behavior: if there is a speed bump on the road for instance, the size of this speed bump should matter. Note that the continuous model is not completely obvious, because, at the level of conservation law, it should involve an information on a set which has a zero Lebesgue measure (the junction point). Our aim is present first a Follow-the-Leader model adapted to this context, then to discuss the continuous models, at the conservation level and at the Hamilton-Jacobi one, and finally to briefly explain how to pass from the micro model to the macro one.

Note that the junction we are studying is very simple: it links one incoming road to one outgoing one, whence the name of a 1:1 junction. We will discuss in the last part the (much more complex) case of p:q junctions, where p is the number of incoming road and q the number of outgoing ones.

3.1 A discrete model

We consider a traffic flow on the line in which again vehicles are ordered and cannot overtake each other. Let $N \in \mathbb{N}$ be the number of vehicles and, for $i \in \{1, \ldots, N\}$, let $X^i(t)$ the position of vehicle labelled i at time t. We assume without loss of generality that $X^i(t) \leqslant X^{i+1}(t)$ for any i and t. In this discrete model, we assume that the vehicles are slowed down at the junction by a speed bump or a traffic light. In this case the equation for the vehicles takes the form

$$\dot{X}^i(t) = W(X^{i+1}(t) - X^i(t), t, X^i(t)), \qquad t \geqslant 0, \ i \in \{1, \ldots, N-1\}. \tag{15}$$

where W is a local perturbation of a fixed velocity rule V: Typically, we assume the following:

$$\begin{array}{ll} 1) & W : [0,\infty) \times \mathbb{R} \times [0,\infty) \to [0,\infty) \text{ is Lipschitz continuous, bounded,} \\ & \text{nondecreasing in the first variable and } T\text{-periodic in time (for some } T > 0), \\ 2) & \text{There exists } e_{\min} > 0 \text{ such that } W(e,t,x) = 0 \text{ for } e \leqslant e_{\min}, \\ 3) & \text{There exists } V : [0,\infty) \to [0,\infty), \text{ Lipschitz continuous and nondecreasing,} \\ & \text{such that } W(e,t,x) = V(e) \text{ if } |x| \geqslant M \text{ and } W \leqslant V \text{ everywhere.} \end{array} \tag{16}$$

Some comments on the assumption are in order: note first that W is a local perturbation of V, which satisfies the hypothesis of the flux of the previous section. We also assume that V is time periodic, having in mind a traffic periodically blocked by a traffic light. The velocity rule V before and after the junction could be different, in order to take into account a change in the configuration of the road. This would not make the problem more difficult (just increase the notation).

As in Proposition 2.1 one can check the existence and the uniqueness of a solution, given an initial condition. Moreover,

$$X^{i+1}(t) \geqslant X^i(t) + e_{\min} \qquad \forall t \geqslant 0,\ \forall i \in \{1,\dots,N\}, \tag{17}$$

provided that this inequality holds at time $t = 0$. Of course, although this is not explicit in the notation, the solution (X^i) depends on the number N of vehicles and on the initial condition.

We set as in Subsection 2.4:

$$\rho^N(t) = \frac{1}{N} \sum_{i \in \{0,\dots,N\}} \delta_{N^{-1}X^i(Nt)}, \qquad t \geqslant 0, \tag{18}$$

The question discussed in this part is the existence of a limit for the ρ^N and its identification. To do so we discuss next the continuous limits models.

3.2 Scalar conservation law on a 1:1 junction: the approach by germs

The limit equation—at the conservation law level—takes the form of our usual LWR model outside the junction $x = 0$, complemented with a condition at the junction. Namely, it reads

$$\begin{cases} \partial_t \rho + \partial_x(f(\rho)) = 0 & \text{in } (0,\infty) \times (\mathbb{R}\backslash\{0\}) \\ (\rho(t,0^-), \rho(t,0^+)) \in \mathcal{G} & \text{in } (0,\infty) \times \{0\} \\ \rho(0,x) = \rho_0(x) & \text{in } \{0\} \times \mathbb{R} \end{cases}$$

Here $(\rho(t,0^-), \rho(t,0^+))$ is the "trace" of ρ at $x = 0$ and the "germ" $\mathcal{G}$ is a subset of $\mathbb{R}_+ \times \mathbb{R}_+$ keeping track of the local perturbation at the junction. We explain these notions next.

3.2.1 Trace, Rankine-Hugoniot condition and dissipation

We fix here $f : [0,R] \to \mathbb{R}_+$ such that $f(0) = f(R) = 0$. We assume that f is Lipschitz continuous and strictly concave in $[0,R]$.

Let us start with the notion of trace. Let ρ be an entropy solution to the scalar conservation law in the half-line

$$\partial_t \rho + \partial_x(f(\rho)) = 0 \qquad \text{in } (0,T) \times (0,\infty).$$

Theorem 3.1. *There exists a map $u \in L^\infty(0,T)$ such that*

$$\text{ess-lim}_{\epsilon \to 0^+} \int_0^T |\rho(t,\epsilon) - u(t)| dt = 0.$$

The limit u is called the trace of ρ on $x = 0$ and denoted by $\rho(\cdot, 0^+)$.

The result holds in a much more general framework and for weaker notions of solutions: see [45, 48]. We refer to these references for a proof. Of course a symmetric result holds on the left-half-space $(-\infty, 0)$, in which case the trace is denoted by $\rho(t, 0^-)$.

Our next aim is now to show first that solutions of the scalar conservation law satisfy natural conditions at (fixed) points of discontinuities. Note carefully that we are interested here in the possibility for the solution to have a shock at a given point for an amount of time of positive measure.

Proposition 3.2. *Assume that $\rho \in L^\infty$ is an entropy solution to*

$$\partial_t \rho + \partial_x(f(\rho)) = 0 \tag{19}$$

in $(0,T) \times (\mathbb{R}\backslash\{0\})$. Then ρ is a weak solution to (19) in $(0,T) \times \mathbb{R}$ if and only if it satisfies the Rankine-Hugoniot condition

$$f(\rho(t,0^-)) = f(\rho(t,0^+)) \qquad \textit{for a.e. } t \in (0,T).$$

For instance, the stationary function $\rho(t,x) = p^-\mathbf{1}_{x<0} + p^+\mathbf{1}_{x>0}$ is a weak solution to the conservation law, if and only if, $f(p^-) = f(p^+)$, because constants are (weak) solutions to (19). On the other hand, let us recall that $\rho(t,x) = p^-\mathbf{1}_{x<\lambda t} + p^+\mathbf{1}_{x>\lambda t}$ is a weak solution to (19), if and only if, $f(p^+) - f(p^-) = \lambda(p^+ - p^-)$.

Proof. We fix a smooth cut-off function $\psi : \mathbb{R} \to [0,1]$ which is equal to 1 outside $(-2,2)$ and vanishes in $[-1,1]$. For $\epsilon > 0$, we set $\psi_\epsilon(x) = \psi(x/\epsilon)$. Let $\phi \in C^1_c((0,\infty) \times \mathbb{R})$. As $\rho \in L^\infty$ is a weak solution to (19) in $(0,T) \times (\mathbb{R}\backslash\{0\})$, we have, using $(t,x) \to \phi(t,x)\psi_\epsilon(t,x)$ as a test function,

$$\int_0^\infty \int_{\mathbb{R}} \rho(t,x)\phi_t(t,x)\psi_\epsilon(t,x) + f(\rho(t,x))(\phi_x(t,x)\psi_\epsilon(t,x) + \phi(t,x)\psi_{\epsilon,x}(x))\ dxdt = 0. \tag{20}$$

Note that, for any $t > 0$,

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{\mathbb{R}} f(\rho(t,x))\phi(t,x)\psi_{\epsilon,x}(x) &= \lim_{\epsilon\to 0}\int_{-2}^2 f(\rho(t,\epsilon y))\phi(t,\epsilon y)\psi_x(y)dy \\
&= \int_{-2}^0 f(\rho(t,0^-))\phi(t,0)\psi_x(y)dy + \int_0^2 f(\rho(t,0^+))\phi(t,0)\psi_x(y)dy \\
&= f(\rho(t,0^+))\phi(t,0) - f(\rho(t,0^-))\phi(t,0),
\end{aligned}$$

since $\psi(0) = 0$ while $\psi(2) = \psi(-2) = 1$. So, as $\epsilon \to 0$, we obtain from (20) that

$$\int_0^\infty \int_{\mathbb{R}} \rho(t,x)\phi_t(t,x) + f(\rho(t,x))\phi_x(t,x)\ dxdt + \int_0^\infty f(\rho(t,0^+))\phi(t,0) - f(\rho(t,0^-))\phi(t,0)\ dt = 0.$$

This implies that, if ρ satisfies the Rankine-Hugoniot condition, then it is a weak solution on $(0,T) \times \mathbb{R}$. Conversely, if ρ is a weak solution on $(0,T) \times \mathbb{R}$, then

$$\int_0^\infty f(\rho(t,0^+))\phi(t,0) - f(\rho(t,0^-))\phi(t,0)\ dt = 0$$

for any test function ϕ. This easily implies that ρ satisfies the Rankine-Hugoniot condition. ☐

Next we investigate the case of an entropy solution. Let us start with a few notation. Recall that f is strictly concave and nonnegative on $[0,R]$. We set $A_{\max} = \max f$ and, for any $A \in [0, A_{\max}]$, we denote by p_A^- and p_A^+ the smallest and the largest solution to $f(p) = A$. Note that $p_A^- < p_A^+$ unless $A = A_{\max}$.

Assume now that ρ is an entropy solution to (19) in $(0,T) \times \mathbb{R}$. By the previous Proposition, we know that the traces of ρ at $x = 0$ satisfies the Rankine-Hugoniot condition. We set

$$A(t) = f(\rho(t,0^-)) = f(\rho(t,0^+)).$$

If the solution ρ has shocks at $x = 0$ and on a set I of positive measure, then we have necessarily $A(t) < A_{\max}$ a.e. in I and either $(\rho(t,0^-),\rho(t,0^+)) = (p^-_{A(t)}, p^+_{A(t)})$, or $(\rho(t,0^-),\rho(t,0^+)) = (p^+_{A(t)}, p^-_{A(t)})$. The next result says that the later equality cannot happen: the solution can only increase through a shock.

Proposition 3.3. *Assume that $\rho \in L^\infty$ is an entropy solution to* (19) *in* $(0,\infty) \times \mathbb{R}$. *We have, for a.e. $t \in (0,T)$, such that $A(t) < A_{\max}$,*

$$(\rho(t,0^-),\rho(t,0^+)) \neq (p^+_{A(t)}, p^-_{A(t)}). \tag{21}$$

Proof. Fix a smooth cut-off function $\psi : \mathbb{R} \to [0,1]$ which is equal to 0 outside $(-2,2)$ and equal to 1 in $[-1,1]$. For $\epsilon > 0$, we set $\psi_\epsilon(x) = \psi(x/\epsilon)$. Let $\phi \in C^1_c((0,\infty), \mathbb{R}_+)$ and $k \geqslant 0$. As $\rho \in L^\infty$ is an entropy solution to (19) in $(0,T) \times \mathbb{R}$, we have, using $(t,x) \to \phi(t,x)\psi_\epsilon(t,x)$ as a test function,

$$\int_0^\infty \int_{\mathbb{R}} |\rho(t,x) - k|\phi'(t)\psi_\epsilon(x) + \mathrm{sgn}(\rho(t,x) - k)(f(\rho(t,x)) - f(k))\phi(t)\psi_{\epsilon,x}(x)\ dxdt \geqslant 0.$$

We can pass to the limit and find, arguing as in the proof of Proposition 3.2 for the term in $\psi_{\epsilon,x}$,

$$-\int_0^\infty (\mathrm{sgn}(\rho(t,0^+)-k)(f(\rho(t,0^+))-f(k)) - \mathrm{sgn}(\rho(t,0^-)-k)(f(\rho(t,0^-))-f(k)))\phi(t)dt \geqslant 0.$$

This proves that, for any $k \geqslant 0$,

$$\mathrm{sgn}(\rho(t,0^+)-k)(f(\rho(t,0^+))-f(k)) - \mathrm{sgn}(\rho(t,0^-)-k)(f(\rho(t,0^-))-f(k))) \leqslant 0 \qquad \text{a.e.}$$

Let $\bar{k}$ be such that $f(\bar{k}) = A_{\max}$. Then the inequality above reads, if $A(t) = f(\rho(t,0^+)) = f(\rho(t,0^-)) < A_{\max}$,

$$\mathrm{sgn}(\rho(t,0^+)-\bar{k}) \geqslant \mathrm{sgn}(\rho(t,0^-)-\bar{k}).$$

This implies that $(\rho(t,0^-),\rho(t,0^+)) \neq (p^+_{A(t)}, p^-_{A(t)})$. ☐

3.2.2 The notion of $\mathcal{G}-$entropic solutions

We fix again a flux function $f : [0,R] \to \mathbb{R}_+$ such that $f(0) = f(R) = 0$. We assume that f is Lipschitz continuous and strictly concave in $[0,R]$ and set $A_{\max} = \max f$. We consider a 1:1 junction, meeting, to fix the ideas, at $x = 0$. We fix a flux limit limiter $\bar{A} \in [0, A_{\max})$: this means that we expect the flux to be not larger then $\bar{A}$ at $x = 0$. This means that the value $\rho(t,0^-) = \rho(t,0^+)$ taken by a solution to our problem at $x = 0$ must satisfy

$$f(\rho(t,0^-)) = f(\rho(t,0^+)) \leqslant \bar{A}.$$

On the other hand, if $f(\rho(t,0^-)) = f(\rho(t,0^+))$ is less than $\bar{A}$, the solution does not see the flux limiter and thus should satisfy condition (21). Hence $(\rho(t,0^-),\rho(t,0^+))$ should take its values in the set

$$\mathcal{G}_{\bar{A}} = \{(p^-,p^+) \in [0,R]^2,\ A := f(p^-) = f(p^+) \leqslant \bar{A}, \text{ and, if } A < \bar{A}, \text{ then } (p^-,p^+) \neq (p^+_A, p^-_A)\}, \qquad (22)$$

where p^-_A and p^+_A are the smallest and the largest solution to $f(p) = A$. Note that $\mathcal{G}_{\bar{A}}$ is a closed set.

The above remark yields to the notion of $\mathcal{G}_{\bar{A}}-$entropy solution for the scalar conservation law

$$\begin{cases} \partial_t \rho + \partial_x (f(\rho)) = 0 & \text{in } (0,\infty) \times (\mathbb{R}\backslash\{0\}) \\ (\rho(t,0^-),\rho(t,0^+)) \in \mathcal{G}_{\bar{A}} & \text{a.e. in } (0,\infty) \times \{0\} \end{cases} \qquad (23)$$

Definition 3.4. *We call a $\mathcal{G}_{\bar{A}}-$entropy solution to* (23) *a map $\rho \in L^\infty((0,\infty)\times\mathbb{R},[0,R])$ is an entropy solution to the scalar conservation law in $(0,\infty)\times(-\infty,0)$ and $(0,\infty)\times(0,\infty)$ and that its trace $(\rho(t,0^-),\rho(t,0^+))$ at $x = 0$ belongs to $\mathcal{G}_{\bar{A}}$ for a.e. $t \geqslant 0$.*

The following result[1] is largely due to [2], which introduces the notion of germ, of $\mathcal{G}_{\bar{A}}-$solution and for uniqueness of this solution; it also prove the contractivity property. The existence of a solution is established in [18], which shows that germs defined by (22) are "good" germs.

Theorem 3.5. *For any initial condition $\rho_0 \in L^\infty(\mathbb{R},[0,R])$, there exists a unique $\mathcal{G}_{\bar{A}}-$solution to* (23) *with initial condition ρ_0. Moreover, the semi-group associated to this evolution equation is mass preserving: if in addition $\rho_0 \in L^1(\mathbb{R})$, then*

$$\int_{\mathbb{R}} \rho(t,x)dx = \int_{\mathbb{R}} \rho_0(x)dx \qquad \forall t \geqslant 0,$$

Moreover the evolution is L^1-contracting: if $\hat{\rho}$ is an another solution to (23) *associated with some initial condition $\hat{\rho}_0 \in L^\infty(\mathbb{R},[0,R])) \cap L^1(\mathbb{R})$, then*

$$\|\hat{\rho}(t) - \rho(t)\|_{L^1} \leqslant \|\hat{\rho}_0 - \rho_0\|_{L^1}.$$

[1] In the context described here, the approach of [23] would also give the result. The notion of germ is especially useful for a discontinuous flux function with a discontinuity at the junction, or more general junctions.

One can check that the only L^1-contracting semi-groups which coincide with the standard conservation law outside $x = 0$ are necessarily $\mathcal{G}_{\bar{A}}-$solutions of a problem of the form (23) for some germ $\mathcal{G}_{\bar{A}}$ given by (22): see [16].

The approach easily generalizes to space-discontinuous fluxes, with a discontinuity at $x = 0$: namely,

$$f(x,p) = f^l(p)\mathbf{1}_{x<0} + f^r(p)\mathbf{1}_{x>0}, \tag{24}$$

for some fluxes f^l and f^r satisfying the same conditions as f. In this setting, the natural condition to be asked on $\bar{A}$ is that $\bar{A} \in [0, A_0]$ where $A_0 = \min\{\max f^l, \max f^r\}$ is the maximal possible flux for f^l and f^r simultaneously. In fact the largest part of the literature dedicated to scalar conservation law on a 1:1 junction is written to handle a discontinuous flux of this form and the lack of uniqueness of the standard entropy solution in this setting. We have decided not to present the argument in this setting for two reasons: the main one is because the notation for discontinuous fluxes are a little heavy and the discontinuity would not bring any additional mathematical difficulty; second because we want to underline that the flux-limiter makes sense even for a space independent flux.

Before proving the theorem, let us start with some preliminary remarks on the germ $\mathcal{G}_{\bar{A}}$.

Lemma 3.6. *The germ $\mathcal{G}_{\bar{A}}$ defined by (22) is dissipative: for any $(c^-, c^+), (d^-, d^+) \in \mathcal{G}_{\bar{A}}$,*

$$q(c^+, d^+) \leqslant q(c^-, d^-),$$

where $q(p,c) = \operatorname{sgn}(p-c)(f(p) - f(c))$. It is also maximal in the sense that, for any $c \in [0,R]^2$,

$$\left[q(c^+, d^+) \leqslant q(c^-, d^-) \qquad \forall d \in \mathcal{G}_{\bar{A}}\right] \qquad \Rightarrow \qquad c \in \mathcal{G}_{\bar{A}}.$$

As we will see below, the first condition ensures the uniqueness of the solution. The maximality implies the L^1-stability of the solution, see [2].

Proof. The proof is tedious but elementary, and consists just in checking the different cases. □

Sketch of proof of Theorem 3.5. The proof of the existence of a $\mathcal{G}_{\bar{A}}-$solution is heavy: it relies on the construction of a scheme to approximate the solution: we refer to [18] for instance.

Let us now explain the proof of the uniqueness. We assume that ρ and $\hat{\rho}$ are two solutions to (23). The key step is Kato's inequality, which reads, for any nonnegative C^1 test function $\xi : [0,\infty) \times \mathbb{R} \to \mathbb{R}_+$ with a compact support,

$$-\int_0^\infty \int_{\mathbb{R}} |\rho(t,x) - \hat{\rho}(t,x)| \partial_t \xi(t,x) + q(\rho(t,x), \hat{\rho}(t,x)) \partial_x \xi(t,x)\ dxdt \leqslant \int_{\mathbb{R}} |\rho(0,x) - \hat{\rho}(0,x)| \xi(0,x)\ dx. \tag{25}$$

One easily derives the contraction property from this new Kato's inequality, exactly as in the standard case.

To prove (25), we fix a smooth cut-off function $\psi : \mathbb{R} \to [0,1]$ which is equal to 1 outside $(-2,2)$ and vanishes in $[-1,1]$. For $\epsilon > 0$, we set $\psi_\epsilon(x) = \psi(x/\epsilon)$. As ρ is a solution to the scalar conservation and as the test function $(t,x) \to \xi(t,x)\psi_\epsilon(x)$ vanishes in a neighborhood of $x = 0$, we have by the standard Kato's inequality (Proposition 2.5):

$$-\int_0^\infty \int_{\mathbb{R}} |\rho(t,x) - \hat{\rho}(t,x)| \partial_t \xi(t,x)\psi_\epsilon(x) + q(\rho(t,x), \hat{\rho}(t,x)) \partial_x(\xi(t,x)\psi_\epsilon(x)) \leqslant \int_{\mathbb{R}} |\rho(0,x) - \hat{\rho}(0,x)| \xi(0,x)\psi_\epsilon(x).$$

As $\epsilon \to 0$, the first and last term converge to the first and last terms in (25). As for the middle one, it splits into the sum $I^1_\epsilon + I^2_\epsilon$, where

$$I^1_\epsilon := \int_0^\infty \int_{\mathbb{R}} q(\rho(t,x), \hat{\rho}(t,x)) \psi_\epsilon(x) \partial_x \xi(t,x),$$

converges to the middle term in (25), and

$$I_\epsilon^2 := \int_0^\infty \int_{\mathbb{R}} q(\rho(t,x), \hat\rho(t,x))\xi(t,x)\partial_x\psi_\epsilon(x) = \int_0^\infty \int_{\mathbb{R}} q(\rho(t,x), \hat\rho(t,x))\xi(t,x)\epsilon^{-1}\psi'(\epsilon^{-1}x).$$

After a change of variable, this last term can be rewritten as

$$I_\epsilon^2 = \int_0^\infty \int_{-\infty}^0 q(\rho(t,\epsilon y), \hat\rho(t,\epsilon y)))\xi(t,\epsilon y)\psi'(y) + \int_0^\infty \int_0^\infty q(\rho(t,\epsilon y), \hat\rho(t,\epsilon y))\xi(t,\epsilon y)\psi'(y).$$

We let $\epsilon \to, 0^+$ and use the notion of trace to get

$$\begin{aligned}\lim I_\epsilon^2 &= \int_0^\infty \int_{-\infty}^0 q(\rho(t,0^-), \hat\rho(t,0^-)))\xi(t,0)\psi'(y) + \int_0^\infty \int_0^\infty q(\rho(t,0^+), \hat\rho(t,0^+))\xi(t,0)\psi'(y) \\ &= \int_0^\infty (-q(\rho(t,0^-), \hat\rho(t,0^-))) + q(\rho(t,0^+), \hat\rho(t,0^+)))\xi(t,0) \ \leqslant 0\end{aligned}$$

by Lemma 3.6 and because $(\rho(t,0^-), \rho(t,0^+)), (\hat\rho(t,0^-), \hat\rho(t,0^+)) \in \mathcal{G}_{\bar A}$. Hence (25) holds. □

3.3 HJ approach: flux limiter

We have seen in the case of a traffic flow on the line that the Hamilton-Jacobi approach to the traffic flow problem provides some useful tools. For this reason, we discuss here the Hamilton-Jacobi counterpart of the notion of $\mathcal{G}_{\bar A}-$solution on a 1:1 junction. The literature on this topic is related to more general questions of Hamilton-Jacobi equations and optimal control problems with discontinuities: we refer to the pioneering works [1, 9, 13, 15, 36, 42] and to the monograph [10]. Unless otherwise stated the results of this part are all borrowed from the work by Imbert and Monneau [36].

We fix a Hamiltonian $f : [0,R] \to \mathbb{R}_+$ as above. We still assume that f is uniformly concave on $[0,R]$ and set $A_{\max} = \max f$. We denote by f^- and f^+ the smallest nonincreasing and nondecrasing functions above f, respectively. Given a flux-limiter $\bar A \in [0, A_{\max}]$, we consider the HJ equation

$$\begin{cases} \partial_t u + f(\partial_x u) = 0 & \text{in } (0,\infty) \times (\mathbb{R}\backslash\{0\}) \\ \partial_t u + F_{\bar A}(\partial_x u(t,0^-), \partial_x u(t,0^+)) = 0 & \text{in } (0,\infty) \times \{0\} \end{cases} \tag{26}$$

where

$$F_{\bar A}(p^-, p^+) = \min\{\bar A,\ f^+(p^-),\ f^-(p^+)\}. \tag{27}$$

The idea is that we want to force the solution u to satisfy $\partial_t u = -\bar A$ at $x = 0$. Indeed, equality $\partial_t u = -\bar A$ at $x = 0$ is the counterpart of the flux condition $f(\rho) = \bar A$ for the conservation law. However this equality is not always possible to satisfy. As explained in [36], the equality $\partial_t u + F_{\bar A}(\partial_x u(t,0^-), \partial_x u(t,0^+)) = 0$ is the "effective" equality corresponding to this condition. Let us underline that the whole part could be treated with a discontinuous Hamiltonian of the form (24) almost without change.

In order to explain the notion of viscosity solution for (26), we introduce the set $PC^1(J)$ of test functions, consisting in the set of continuous maps $\phi : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ such that the restrictions $\phi_{(0,\infty)\times|_{(-\infty,0]}}$ and $\phi_{(0,\infty)\times|_{[0,\infty)}}$ are C^1. Note that such a map has not necessarily a space derivative at $(t,0)$, but has a left- and a right- space derivatives at this point, denoted by $\partial_x\phi(t,0^-)$ and $\partial_x\phi(t,0^+)$ respectively. On the other hand, the time derivative $\partial_t\phi(t,0)$ exists and is continuous at this point.

Definition 3.7. *We say that $u : [0,\infty \times \mathbb{R} \to \mathbb{R}$ is a (flux-limited) viscosity subsolution of* (26) *if u is Lipschitz continuous in $[0,\infty) \times \mathbb{R}$, with $\partial_x u \in [0,R]$ a.e., if u is a viscosity subsolution of the HJ equation in $(0,\infty) \times (\mathbb{R}\backslash\{0\})$ (in the standard sense, see Definition 2.6) and if, for any test function $\phi \in PC^1(J)$ such that $u - \phi$ has a local maximum at some point $(t,0)$ with $t > 0$, then*

$$\partial_t\phi(t,0) + F_{\bar A}(\partial_x\phi(t,0^-), \partial_x\phi(t,0^+)) \leqslant 0.$$

A symmetric definition holds for supersolution. A solution is a map which is at the same time sub- and supersolution.

A few remarks are now in order. First we explain that the notion of supersolution at $x = 0$ can be understood almost in a classical sense:

Proposition 3.8. *Let u be Lipschitz continuous in $[0,\infty) \times \mathbb{R}$, with $\partial_x u \in [0,R]$ a.e., and such that u is a viscosity subsolution of the HJ equation in $(0,\infty) \times (\mathbb{R}\backslash\{0\})$. Then u is a flux-limited viscosity supersolution to (26), if and only if, for a.e. $t > 0$ and in the viscosity sense,*

$$\partial_t u(t,0) + \bar{A} \geqslant 0. \tag{28}$$

Remark 3.9. Recall that, at least formally, $\partial_t u(t,0) = -f(\partial_x u(t,0^+)) = -f(\partial_x \rho(t,0^-))$, so that (28) indeed says that the flux is limited by $\bar{A}$ at $x = 0$.

Proof. It is known that, for a linear inequality like (28) and a Lipschitz map like $t \to u(t,\cdot)$, this inequality holds a.e. if and only if it holds in the viscosity sense (see [37]).

Let us first assume that u is viscosity supersolution and let us check that (28) holds. Let $\phi = \phi(t)$ be a smooth test function such that $u - \phi$ has a minimum at $\bar{t} \in (0,\infty)$. Without loss of generality, we assume that $\phi(\bar{t}) = u(\bar{t},0)$. As $\partial_x u \in [0,R]$, we have therefore

$$u(t,x) \geqslant u(t,0) + Rx\mathbf{1}_{x<0} \geqslant \phi(t) + Rx\mathbf{1}_{x<0},$$

with an equality at $(\bar{t},0)$. Using the fact that u is viscosity supersolution, we infer that

$$0 \leqslant \phi'(t) + \min\{\bar{A}, f^+(R), f^-(0)) = \phi'(t) + \bar{A},$$

since $f^+(R) = f^-(0) = \max f$ and $\bar{A} \in [0, \max f]$.

Conversely, assume that u is a viscosity solution in $(0,\infty) \times (\mathbb{R}\backslash\{0\})$ and satisfies (28) in the viscosity sense. Let $\phi \in PC^1(J)$ be a test function and assume that $u - \phi$ has a strict local minimum at $(\bar{t},0)$, with $\bar{t} > 0$. We are first going to check that

$$\partial_t \phi(\bar{t},0) + f^-(\phi(\bar{t},0^+)) \geqslant 0. \tag{29}$$

We will see that this inequality holds just because u is a supersolution to the HJ equation in $(0,\infty)^2$. For this we argue by penalization. For $\epsilon > 0$ small, the map $u - \phi + \epsilon/x$ has a local minimum at (t_ϵ, x_ϵ), with $(t_\epsilon, x_\epsilon) \to (\bar{t},0)$ as $\epsilon \to 0$. Thus, as u is a supersolution to the HJ equation in $(0,\infty)^2$, we infer that

$$\partial_t \phi(t_\epsilon, x_\epsilon) + f(\partial_x \phi(t_\epsilon, x_\epsilon) + \frac{\epsilon}{x_\epsilon^2}) \geqslant 0.$$

But $f \leqslant f^-$ and f^- is nonincreasing while the term $\frac{\epsilon}{x^2}$ is positive. Thus

$$\partial_t \phi(t_\epsilon, x_\epsilon) + f^-(\partial_x \phi(t_\epsilon, x_\epsilon)) \geqslant 0.$$

Letting $\epsilon \to 0^+$ gives (29). We obtain in a similar way the inequality

$$\partial_t \phi(\bar{t},0) + f^+(\phi(\bar{t},0^-)) \geqslant 0.$$

Finally, we note that $t \to u(t,0) - \phi(t,0)$ has a local minimum at $\bar{t}$. Hence by (28) we infer that $\partial_t \phi(\bar{t},0) + \bar{A} \geqslant 0$. In conclusion, we have proved that

$$\partial_t \phi(\bar{t},0) + \min\{\bar{A}, f^+(\phi(\bar{t},0^-), f^-(\phi(\bar{t},0^+))\} \geqslant 0,$$

which shows that u is a viscosity supersolution. □

Next we explain that the set of test functions at the junction can be reduced drastically for subsolutions.

Proposition 3.10 (Reduced test function)**.** *Let u be a Lipschitz continuous viscosity subsolution to the HJ equation in $(0,\infty) \times (\mathbb{R}\backslash\{0\})$. Then u is a viscosity subsolution to (26), if and only if, for any test function of the form $(t,x) \to \phi(x) + \psi(t)$ belonging to $PC^1(J)$, such that $f(\partial_x \phi(0^-)) = f(\partial_x \phi(0^+)) = \bar{A}$, if $u - \phi$ has a local maximum at some point $(t,0)$ with $t > 0$, then*

$$\psi'(t) + \bar{A} \leqslant 0.$$

Note that Proposition 3.8 says that a much stronger statement holds for supersolutions. The proof is technical and can be found in [36] for instance. The key result on viscosity solution to (26) is the following comparison principle, the proof of which also exceeds the scope of these short notes (see [36], and also different arguments in [10, 42]).

Theorem 3.11 (Comparison)**.** *Let u and v be respectively a subsolution and a supersolution to* (26)*. If $u(0,\cdot) \leqslant v(0,\cdot)$ in $\mathbb{R}$, then $u \leqslant v$ in $[0,\infty) \times \mathbb{R}$.*

As a consequence, we have

Theorem 3.12 (Existence and uniqueness)**.** *Given a Lipschitz initial condition $u_0 : \mathbb{R} \to \mathbb{R}$ with $u_0' \in [0, R]$ a.e., there exists a unique viscosity solution to* (26) *with initial condition u_0.*

As for CL/HJ on the line, there is a strong relationship between $\mathcal{G}_{\bar{A}}-$entropy solutions to (23) and flux limited viscosity solutions to (26).

Theorem 3.13 (From HJ to CL for 1:1 junctions [18])**.** *Fix $A \in [0, A_{\max}]$. If $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a Lipschitz map such that $\partial_x u \in [0, R]$ a.e. is a viscosity solution to* (26) *with $F_{\bar{A}}$ defined in* (27)*, then $\rho = \partial_x u$ is a $\mathcal{G}_{\bar{A}}-$entropy solution to* (23) *with a germ given by* (22)*.*

Sketch of proof. Let $\rho = \partial_x u$. We know from Proposition 2.9 that ρ is an entropy solution to the CL in $(0,\infty) \times (\mathbb{R}\backslash\{0\})$. It remains to check that the trace $(\rho(t,0^-), \rho(t,0^+))$ belongs to $\mathcal{G}_{\bar{A}}$.

We first claim that the Rankine-Hugoniot equality $f(\rho(t,0^-)) = f(\rho(t,0^+)) = -\partial_t u(t,0)$ holds a.e.. For any $\xi \in C_c^\infty((0,+\infty))$ and $h > 0$ small, we have, after integrating the equation of u which is satisfied a.e.

$$h^{-1} \int_0^\infty \int_0^h \xi(t) f(\rho(t,x))\ dxdt = h^{-1} \int_0^\infty \int_0^h \xi'(t) u(t,x)\ dxdt.$$

By continuity of u, the right-hand side converges, as $h \to 0^+$, to $\int_0^\infty \xi'(t)u(t,0)dt$. By the strong trace property, the left-hand side converges to $\int_0^\infty \xi(t) f(\rho(t,0^+))\ dt$ as $h \to 0^+$. This implies that

$$\int_0^\infty \xi(t) f(\rho(t,0^+))\ dt = \int_0^\infty \xi'(t)u(t,0)dt = -\int_0^\infty \xi(t)\partial_t u(t,0)dt.$$

In the same way, we have

$$\int_0^\infty \xi(t) f(\rho(t,0^-))\ dt = \int_0^\infty \xi'(t)u(t,0)dt,$$

which completes the proof of the claim.

We also know by Proposition 3.8 that $-\partial_t u(t,0) \leqslant \bar{A}$, so that $f(\rho(t,0^-)) = f(\rho(t,0^+)) \leqslant \bar{A}$ a.e.. In addition, in view of the existence of right- and left- derivatives of $u(t,\cdot)$ at points $(t,0)$ where $\partial_t u(t,0)$ exists [43], one can easily check that $\rho(t,0^\pm) = \partial_x u(t,0^\pm)$. It remains to check that $(\partial_x u(t,0^-), \partial_x u(t,0^+)) \in \mathcal{G}_{\bar{A}}$.

To prove the claim, we argue by contradiction, assuming that, for some $t > 0$ at which $\partial_t u(t,0)$ exists (and thus $\partial_x u(t,0^-)$ and $\partial_x u(t,0^+)$ exist as well),

$$f(\rho(t,0^-)) < A, \quad (\rho(t,0^-), \rho(t,0^+)) = (p_{\bar{A}}^+, p_{\bar{A}}^-).$$

Let us fix $\epsilon > 0$ so small that $\lambda := f(\rho(t,0^-)) + \epsilon < \bar{A}$. As f is concave, we have $p_\lambda^+ < \partial_x u(t,0^-)$ and $p_\lambda^- > \partial_x u(t,0^+)$. Let us define the map $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$w(s,x) = u(t,0) + \begin{cases} p_\lambda^+ x - \lambda s & \text{if } x \leqslant 0 \\ p_\lambda^- x - \lambda s & \text{if } x \geqslant 0 \end{cases}$$

Then w is a test function which is a supersolution of the HJ equation because,

$$f(p_\lambda^+) = f(p_\lambda^-) = \lambda = -\partial_s w$$

and using $\bar{A} \in [0, \max f]$, we get

$$\min\{\bar{A}, f^+(p_\lambda^+), f^-(p_\lambda^+)\} = \min\{\bar{A}, \max f, \max f\} = \bar{A} \geqslant \lambda = -\partial_s w.$$

Moreover, we have that $u(t,x) \leqslant w(0,x)$ if $|x|$ is small enough. Thus, by finite speed of propagation and comparison, we have $u(t+h,0) \leqslant w(h,0)$ for $h>0$ small enough. Therefore

$$-f(\rho(t,0^-)) = \partial_t u(t,0) \leqslant \partial_s w(0,0) = -\lambda = -f(\rho(t,0^-)) - \epsilon,$$

which is impossible. This proves that $(\rho(t,0^-),\rho(t,0^+)) \in \mathcal{G}_{\bar{A}}$.

□

3.4 From micro to macro

We now go back to discrete dynamic $(X^i)_{i=1,\dots,N}$ defined in Subsection 3.1 (with $\dot{X}^N = \max V$) and define the flow of measures ρ^N by (18). The following result is due to [30].

Theorem 3.14 (Micro-Macro derivation for 1:1 junctions)**.** *Under Assumption* (16)*, there exists a flux limiter $\bar{A} \in [0, \max f]$ such that, if condition* (17) *holds and if $\rho^N(0)$ converges weakly to ρ_0, then the flow of measures (ρ^N) converges weakly-* to the $\mathcal{G}_{\bar{A}}-$entropic solution to* (23)*.*

The proof in [30] consists in transforming the discrete system into a (nonlocal) HJ equation, and then use argument in homogenization of HJ equation [41] (here, construction of a local corrector) to pass to the limit at the level of the flux-limited HJ equation. The conclusion then follows from Theorem 3.13 on the relationship between $\mathcal{G}_{\bar{A}}-$solutions of (23) and flux-limited viscosity solutions of HJ equation (26).

We sketch here a different approach, which will be useful in the case of different junctions (Section 4.1.2). It also relies on Theorem 3.13 but consists in building directly the flux-limitor.

Let us first mention as a starting point a kind of localization/comparison principle:

Proposition 3.15 (Discrete comparison principle, [20])**.** *Let $(\tilde{X}^i)_{i=1,\dots,N}$ be another solution to* (15) *and assume that $X^i(0) \leqslant \tilde{X}^i(0)$ for any $i \in \{1,\dots,N\}$. Then there exists $\beta>0$ and $C>0$ depending on W only such that, for any $i \in \{1,\dots,N\}$ and any $t \geqslant 0$,*

$$X^i(t) \leqslant \tilde{X}^i(t) + C2^{-(N-i)}e^{\beta t}.$$

The result is interesting, for large time intervals, when the number of vehicles is large as well. It is then a mixture between a comparison principle and a finite speed of propagation property. In particular, when there are infinitely many vehicles, we have a full comparison principle: if $(X^i)_{i\in\mathbb{N}}$ and $(\tilde{X}^i)_{i\in\mathbb{N}}$ are two solutions of (15) with $N=\infty$, and if $X^i(0) \leqslant \tilde{X}^i(0)$ for any $i \in \mathbb{N}$, then $X^i(t) \leqslant \tilde{X}^i(t)$ for any $t \geqslant 0$ and any $i \in \mathbb{N}$.

The central argument of the proof of Theorem 3.14 consists in solving the problem in a particular case. Let $\bar{e}>0$ be such that $f_0(1/\bar{e}) = \max f_0$. We consider the solution to

$$\dot{Y}^i(t) = V(Y^{i+1}(t) - Y^i(t), t, Y^i(t)), \qquad t \geqslant 0,\ i \in \mathbb{Z}, \tag{30}$$

with initial condition

$$Y^i(0) = i\bar{e}, \qquad \forall i \in \mathbb{Z}.$$

It is not difficult to check that the solution exists for all time and satisfies

$$Y^{i+1}(t) \geqslant Y^i(t) + e_{\min} \qquad \forall i \in \mathbb{Z}. \tag{31}$$

We introduce

$$\rho^N_Y(t) = N^{-1}\sum_{i\in\mathbb{Z}} \delta_{N^{-1}Y^i(Nt)}$$

and

$$u^N_Y(t,x) = N^{-1}\sum_{i>0} \mathbf{1}_{\{x>N^{-1}Y^i(Nt)\}} - N^{-1}\sum_{i\leqslant 0} \mathbf{1}_{\{x\leqslant N^{-1}Y^i(Nt)\}}$$

We note that $\partial_x u^N_Y = \rho^N_Y$ in the sense of distribution. Following arguments similar to the ones in the proof of Theorem 2.10, it is not difficult to check that the family (u^N_Y) is equicontinuous.

Proposition 3.16. *There exists $\bar{A} \in [0, A_{\max}]$ such that*

$$u_Y(t,x) := \lim_N u_Y^N(t,x) = \max\left\{x/\bar{e} - tf_{\max}, p_{\bar{A}}^+ x\mathbf{1}_{\{x<0\}} + p_{\bar{A}}^- x\mathbf{1}_{\{x>0\}} - t\bar{A}\right\},$$

locally uniformly in $[0,\infty) \times \mathbb{R}$. In particular, the limit u_Y is a viscosity solution to the flux-limited HJ equation (26).

The key point is that, in a neighborhood of $x = 0$, the right-hand side equals $p_{\bar{A}}^+ x\mathbf{1}_{\{x<0\}} + p_{\bar{A}}^- x\mathbf{1}_{\{x>0\}} - t\bar{A}$, which is a self-similar viscosity solution to the flux limited HJ equation (26). Thus we will be able to use the map u_Y^N as a kind of corrector.

The proof of the proposition requires several steps. We consider the quantity

$$m(t) = \max\{m \in \mathbb{R},\ \forall i \in \mathbb{Z},\ Y^i(t) \geqslant i\bar{e} + m\} = \inf_{i\in\mathbb{Z}} Y^i(t) - i\bar{e}.$$

As the Y^i are nondecreasing and uniformly Lipschitz continuous, $m(\cdot)$ is nondecreasing and Lipschitz continuous. Note that, if the dynamic was simply given by V, then we would have $Y^i(t) = i\bar{e} + tV(\bar{e})$ and thus we would have $m(t) = tV(\bar{e})$. Because $W \leqslant V$, one can check that $Y^i(t) \leqslant i\bar{e} + tV(\bar{e})$ for any i. Thus $m(t) \leqslant tV(\bar{e})$. We claim that

Lemma 3.17. *The limit* $\bar{N} := \lim_{t\to\infty} \dfrac{N(t)}{t}$ *exists and belongs to* $[0, A_{\max}]$.

Proof. Fix $n, p \in \mathbb{N}$. Then

$$Y^i(nT) \geqslant (i + [m(nT)/\bar{e}])\bar{e} \qquad \forall i \in \mathbb{Z}.$$

Let $(Z^i(s))_{i\in\mathbb{Z}}$ be the solution of (30) starting at $Z^i(0) = (i + [m(nT)/\bar{e}])\bar{e}$. By the periodicity of V in time and the definition of $m(s)$, we have

$$Z^i((n+p)T) = Y^{i+[m(nT)/\bar{e}]}(pT) \geqslant (i + [m(nT)/\bar{e}])\bar{e} + m(pT) \qquad \forall i \in \mathbb{Z}.$$

On the other hand, by discrete comparison (Proposition 3.15 with $N = \infty$), we have $Y^i(nT + s) \geqslant Z^i(s)$ for any s. Hence

$$m((n+p)T) \geqslant [m(nT)/\bar{e}]\bar{e} + m(pT) \geqslant m(nT) + m(pT) - 1/\bar{e}.$$

This shows that $\tilde{m}(n) = m(nT) - 1/\bar{e}$ satisfies the superadditive property:

$$\tilde{m}(n+p) \geqslant \tilde{m}(n) + \tilde{m}(p).$$

By Fekete's Lemma, we infer that the following limits exist:

$$\lim_n \frac{N(nT)}{n} = \lim_n \frac{\tilde{m}(n)}{n} = \inf_n \frac{\tilde{m}(n)}{n}.$$

The map N being globally Lipschitz, the result follows. □

The next step consists in showing that $\bar{N}$ is roughly the decay rate of $u_Y^N(t, 0)$:

Lemma 3.18. *We have*

$$\lim_{N\to\infty} u_Y^N(t,0) = -\frac{\bar{N}}{\bar{e}}t.$$

Sketch of proof. Note first that

$$u_Y^N(t,0) = -N^{-1} \sum_{i\leqslant 0} \mathbf{1}_{\{0\leqslant N^{-1}Y^i(Nt)\}}$$

By the definition of $m(t)$, we have

$$Y^i(t) \geqslant i\bar{e} + m(t) \qquad \forall i \in \mathbb{Z}.$$

Hence
$$u_Y^N(t,0) \geqslant -N^{-1}\sum_{i\leqslant 0} \mathbf{1}_{\{0>N^{-1}(i\bar{e}+m(Nt))\}} \geqslant -N^{-1}\bar{e}^{-1}(m(Nt)+1).$$
This implies that
$$\liminf_N u_Y^N(t,0) \geqslant -t\frac{\bar{N}}{\bar{e}}.$$
To obtain the lower bound, fix $t > 0$. We show below in Lemma 3.19 that, at least under suitable conditions, there exists $i_0 \in \mathbb{Z}$ such that $Y^{i_0}(Nt) = i_0\bar{e} + m(Nt)$ and $Y^{i_0}(Nt) \in [-M, M]$. Note that
$$i_0 \geqslant -\bar{e}^{-1}(M + m(Nt)).$$
Moroever, for $i \geqslant i_0 + M/e_{\min}$, we have by (31),
$$Y^i(Nt) \geqslant Y^{i_0}(Nt) + M \geqslant 0,$$
so that, for some constant $C > 0$,
$$u_Y^N(t,0) \leqslant -N^{-1}\sum_{i\leqslant 0} \mathbf{1}_{\{i\geqslant i_0+M\delta\}} \leqslant -N^{-1}\bar{e}^{-1}(m(Nt)+C).$$
Letting $N \to \infty$, we obtain
$$\limsup_N u_Y^N(t,0) \leqslant -\frac{\bar{N}}{\bar{e}}t.$$
In practice, the conditions required by Lemma 3.19 below are not always met and one has to consider a slightly different quantity than $m(t)$ to conclude (see the argument in [20]). ☐

Lemma 3.19. *Fix $t > 0$ and assume that $m(t) < V(\bar{e})t$ and that $m'(t)$ exists and satisfies $m'(t) < V(\bar{e})$. Then there exists $i_0 \in \mathbb{Z}$ such that $Y^{i_0}(t) \in [-M, M]$ and $m(t) = Y^{i_0}(t) - \bar{e}i_0$.*

In practice, the assumption $m(t) < V(\bar{e})t$ holds in the interesting situation where $\bar{N} < V(\bar{e})$ and t is large. Then second condition, is then "often" satisfied. Note that $-i_0$ (which is positive for t large) can be interpreted as the number of vehicles having crossed the junction on the time interval $[0, t]$.

Proof. By Proposition 3.15, one easily checks that
$$\lim_{|i|\to\infty} Y^i(t) - (\bar{e}i + V(\bar{e})t) = 0.$$
Thus, as $m(t) < V(\bar{e})t$, the infimum defining $m(t)$ is reached at some $i_0 \in \mathbb{Z}$: $m(t) = Y^{i_0}(t) - \bar{e}i_0$. This implies that
$$Y^{i_0+1}(t) - \bar{e}(i_0+1) \geqslant Y^{i_0}(t) - \bar{e}i_0,$$
and thus
$$Y^{i_0+1}(t) - Y^{i_0}(t) \geqslant \bar{e}.$$
Using the Envelope Theorem, the fact that V is nondecreasing in its first variable and the previous inequality, we then have
$$m'(t) = \dot{Y}^{i_0}(t) = W(Y^{i_0+1}(t) - Y^{i_0}(t), Y^{i_0}(t)) \geqslant W(\bar{e}, Y^{i_0}(t)).$$
On the other hand we have assumed that $m'(t) < V(\bar{e})$, so that by assumption (16)-(3), we must have $Y^{i_0}(t) \in [-M, M]$. ☐

Proof of Proposition 3.16. Let u be the uniform limit of the u^N, up to a subsequence. Note that, by the definition of $Y^i(0)$, we have $u(0,x) = x/\bar{e}$. Following the same argument as in the proof of Theorem 2.10 we can show that u is a solution to the HJ equation in $(0,\infty)\times(\mathbb{R}\backslash\{0\})$. In addition, by Lemma 3.18, we know that $u(t,0) = -\bar{A}t$ for any $t \geqslant 0$. Thus u is the unique viscosity solution to the boundary problem
$$\begin{cases} \partial_t u + f(\partial_x u) = 0 & \text{in } (0,\infty)\times(\mathbb{R}\backslash\{0\}) \\ u(t,0) = -\bar{A}t & \text{in } (0,\infty) \\ u(0,x) = x/\bar{e} & \text{in } \mathbb{R} \end{cases}$$

As the solution to this equation is unique, we infer that the whole sequence (u^N) converge to this limit. We finally check that u is given by

$$u(t,x) = \max\left\{x/\bar{e} - tf_{\max}, p_{\bar{A}}^+ x\mathbf{1}_{\{x<0\}} + p_{\bar{A}}^- x\mathbf{1}_{\{x>0\}} - t\bar{A}\right\}.$$

Indeed, let $v = v(t,x)$ denote the right-hand side of this equality. Note that the maps $(t,x) \to x/\bar{e} - tf_{\max}$ and $(t,x) \to p_{\bar{A}}^+ x\mathbf{1}_{\{x<0\}} + p_{\bar{A}}^- x\mathbf{1}_{\{x>0\}} - t\bar{A}$ are both solutions to the HJ equation in $(0,\infty) \times (\mathbb{R} \times \{0\})$. As the Hamiltonian f is concave, it is not difficult to check that the supremum of two viscosity solutions is sill a viscosity solution. Thus v solves the HJ equation in $(0,\infty) \times (\mathbb{R} \times \{0\})$. On the other hand, $v(0,x) = x/\bar{e} - tf_{\max}$ while $v(t,0) = -\bar{A}$. This proves that $v = u$. To prove that v is a viscosity solution to (26), we just need to check the condition at $x = 0$, which is satisfied in the classical sense because:

$$\partial_t v(t,0) + \min\{\bar{A}, f^-(\partial_x v(t,0^-)), f^+(\partial_x v(t,0^+))\} = -\bar{A} + \min\{\bar{A}, f^-(p_{\bar{A}}^+), f^+(p_{\bar{A}}^-)\} = -\bar{A} + \bar{A} = 0$$

since $f^-(p_{\bar{A}}^+) = f^+(p_{\bar{A}}^-) = \bar{A}$. □

To explain the role of the "corrector" built in Proposition 3.16, we now explain how to use it to prove the supersolution property. The proof of the subsolution property can be obtained by similar (but more technical) argument, using Proposition 3.10 in the place of Proposition 3.8.

Rough sketch of proof of Theorem 3.14: supersolution. Let X^i be the solution to (15) with initial condition satisfying (17). Let us define

$$u^N(t,x) = N^{-1}\sum_{i=1}^{N} \mathbf{1}_{\{x>N^{-1}X^i(Nt)\}}$$

One can check, exactly as in the proof of Theorem 2.10, that u^N is equicontinuous and converges, up to a subsequence denoted in the same way, to some u where u solves the HJ equation in $(0,\infty)\times(\mathbb{R}\backslash\{0\})$. Let us prove that u is a supersolution at $x = 0$: for this we need to check that $\partial_t u + \bar{A} \geqslant 0$ a.e. (Proposition 3.8). By the Rankine-Hugoniot (see for instance the argument in the proof of Theorem **??**), we have

$$\partial_t u(t,0) = -f(\partial_x u(t,0^-)) = -f(\partial_x u(t,0^+)) \qquad \text{a.e.}.$$

Suppose that $(\bar{t},0)$ is a point of "semi-differentiability" of u such that $\partial_t u(\bar{t},0) < -\bar{A}$ (it is proved in [43] that such a point exists a.e.). Let $A := f(\partial_x u(t,0^-)) = f(\partial_x u(t,0^+)) > 0$. Then

$$u(\bar{t},x) \geqslant u(\bar{t},0) + p_{\bar{A}}^+ x\mathbf{1}_{x\leqslant 0} + p_{\bar{A}}^- x\mathbf{1}_{x\geqslant 0} + |x|\epsilon(x) \ \geqslant\ u(\bar{t},0) + p_{\bar{A}}^+ x\mathbf{1}_{x\leqslant 0} + p_{\bar{A}}^- x\mathbf{1}_{x\geqslant 0}$$

in a neighborhood of $x = 0$. Using Proposition 3.16, this implies that, for N large enough,

$$u^N(\bar{t},x) - u_Y^N(\bar{t},x) \geqslant u(\bar{t},0) + \bar{A}\bar{t} + o_N(1)$$

in a neighborhood of $x = 0$. Using the local comparison argument in Proposition 3.15 (after unscaling), we infer that, for $h > 0$ small,

$$u^N(\bar{t}+h,0) - u_Y^N(\bar{t}+h,0) \geqslant u(\bar{t},0) + \bar{A}t + o_N(1).$$

Letting $N \to \infty$, we infer (again by Proposition 3.16) that

$$u(\bar{t}+h,0) + \bar{A}(\bar{t}+h) \geqslant u(\bar{t},0) + \bar{A}\bar{t},$$

which is impossible since $\partial_t u(\bar{t},0) < -\bar{A}$ by assumption. □

4 Traffic flow on more complex junctions

The analysis of traffic flows on junctions involving more than two branches and the corresponding micro-macro derivation is largely an open question and is a much more challenging than the 1:1 case. At a micro level, it is easy to understand that the different priority rules introduce a huge variety of possible models. In the few example which have been treated in details so far, the variety still holds at the continuous level. We will present here models involving 3 half-lines: either a bifurcation—with a single entry line and two exit lines (the 1:2 model), or the merging of two entry lines into a single one (the 2:1 model).

4.1 A model of a bifurcation: approach by HJ

We concentrate here on a bifurcation: it is a network involving a single entry line and several exit lines. To fix the ideas and for simplicity we discuss the case of two exit roads, say Road 1 and Road 2, but the general case could be handled exactly in the same way. Our goal is to describe discrete and continuous models in which a given and fixed proportion of the vehicles of the entry line goes into Road 1 (the other entering Road 2). It turns out that the HJ formalism—and the notion of flux limiter—is adapted to this setting. This part is largely borrowed from [19].

4.1.1 The HJ formulation of the continuous model

Let $\mathcal{R}^0 = (0, \infty) \times \{0\}$ be the outgoing branch, $\mathcal{R}^j = (-\infty, 0) \times \{j\}$ for $j = 1, 2$ being the incoming ones. The set $\mathcal{R} = \bigcup_{j=0}^{2} \mathcal{R}^j \cup \{0\}$ is endowed with the topology of three half lines glued together at the origin 0. We look for densities $(\rho^k)_{k\in\{0,1,2\}}$ defined on $\mathcal{R}$, satisfying the usual conservation law in the interior of each branch and such that, at the junction,

$$\rho^k(t, 0^+) = \pi^k \rho^0(t, 0^-)$$

where $\pi^k \in (0, 1)$ is the fixed proportion of vehicle entering branch k (where $k = 1, 2$). Note that $\pi^1 + \pi^2 = 1$. It turns out that the germ

$$\mathcal{G} := \{(q^0, q^1, q^2) \in [0, R]^3,\ f(q^0) = f(q^1) + f(q^2),\ q^k = \pi^k q^0 \text{ for } k = 1, 2\}$$

does not satisfy the desired dissipativity property (recall Lemma 3.6). No well-posedness theory is known for this germ. It turns out that there is a good theory for the integrated form of the equation. Namely, let us consider the system of HJ equations with continuous unknown $(u^k)_{k=0,1,2}$. Here u^0 is an antiderivative of ρ^0 but, for technical reasons, u^k is an antiderivative of ρ^k/π^k for $k = 1, 2$. We assume that the u^k are continuous up to the junction: $u^0(t, 0^-) = u^1(t, 0^+) = u^2(t, 0^+)$, and satisfy

$$\begin{cases} \partial_t u^0 + f^0(\partial_x u^k) = 0 & \text{in } (0, \infty) \times (-\infty, 0) \\ \partial_t u^k + f^k(\partial_x u^k) = 0 & \text{in } (0, \infty) \times (0, \infty),\ k = 1, 2 \\ \partial_t u + F_{\bar{A}}(\partial_x u^0(t, 0^-), \partial_x u^1(t, 0^+), \partial_x u^2(t, 0^+)) = 0 & \text{in } (0, \infty) \times \{0\} \end{cases} \tag{32}$$

Above,

$$f^0 = f, \qquad f^k(p) = \frac{1}{\pi^k} f(\pi^k p) \text{ for } k = 1, 2,$$

and

$$F_{\bar{A}}(p^0, p^1, p^2) = \min\{\bar{A},\ f^{0,+}(p^0),\ f^{1,-}(p^1),\ f^{2,-}(p^2)\}$$

for some flux limiter $\bar{A} \in [0, A_0]$, where $A_0 = \min_k \max_p f^k(p)$. As in Section 3.3, for a given function g, g^- (resp. g^+) denotes the smallest nonincreasing (resp. nondecreasing) function above g.

To define the notion of viscosity solution for (32), we introduce the set of test function $PC^1(J)$. It is the set of maps $\phi = (\phi^k)_{k=0,1,2}$ which are C^1 on $(0, \infty) \times (-\infty, 0]$ for ϕ^0 and on $(0, \infty) \times [0, \infty)$ for ϕ^k $(k = 1, 2)$, and such that $\phi^0(t, 0^-) = \phi^1(t, 0^+) = \phi^2(t, 0^+)$. The notion of viscosity solution in this context is exactly the same as in Definition 3.7 and the existence and uniqueness of a solution, given an initial condition $u_0 = (u_0^k)$ hold as in Theorem 3.12, following a comparison result similar to the one in Theorem 3.11 (see [36]).

4.1.2 Micro to macro derivation

In the microscopic model, there are N vehicles. A vehicle evolves first on road 0 before entering either road 1 or road 2. The road it eventualy enters is determined from the initial time and does not change with time. Exactly as in Section 2.5.2, it can be interpreted as a "type" Z_i associated with vehicle i: $Z_i \in \{1, 2\}$ equals 1 if vehicle i intends to enter road 1 and $Z_i = 2$ otherwise. We assume that the (Z_i) are independent, with $\mathbb{P}[Z_i = k] = \pi^k$.

The dynamics of vehicle i is governed by its type, and the distance of the vehicle in front of it. In contrast with the case of 1:1 junctions, the vehicle in front of car i might change in time (before and after the junction). We denote by $i+1$ the label of the vehicle in front of car i before the junction, and by ℓ_i the label of the vehicle in front of car i after the junction. Note that this later vehicle is the first one in front of vehicle i and which is of the same type as i. Thus its label is determined from the initial time and does not change in time. This yields to a dynamics of the form

$$\dot{X}^i(t) = V_{Z_i}(X^{i+1}(t) - X^i(t), X^{\ell_i}(t) - X^i(t), X^i(t)).$$

The map $V : \{1,2\} \times [0,\infty)^2 \times \mathbb{R} \to [0,\infty)$ is supposed to Lipschitz continuous. As in Section 3.1, it is reasonable to assume that V depends only on $X^{i+1}(t) - X^i(t)$ far before the junction, and only on $X^{\ell_i}(t) - X^i(t)$ far after the junction: namely we suppose that, for some $M > 0$,

$$V_z(e^0, e^1, x) = \begin{cases} V(e^0) & \text{if } x \leqslant -M, \\ V(e^1) & \text{if } x \geqslant M. \end{cases}$$

The transition zone $[-M, M]$ corresponds to a portion of space in which a vehicle has to take into account not only the vehicle in front before the junction, but also the vehicle which will be in front after the junction.

Let us consider

$$\rho^{N,0}(t, dx) = N^{-1} \sum_{i \in \{1,\dots,N\},\ X^i(Nt) < 0} \delta_{N^{-1}X^i(Nt)}$$

and, for $k = 1, 2$,

$$\rho^{N,k}(t, dx) = N^{-1} \sum_{i \in \{1,\dots,N\},\ X^i(Nt) \geqslant 0,\ Z_i = k} \delta_{N^{-1}X^i(Nt)}.$$

We have the following convergence result [20] (see also [31] for a deterministic model):

Theorem 4.1. *Under suitable assumption on V, there exists a flux limiter $\bar{A}$ such that the following holds: for any suitable the initial condition $(\rho^{N,k}(0))$ converging weakly to densities $\rho_0 = (\rho_0^k)$, the $(\rho^{N,k})$ defined above converge weakly to (ρ^k) as $N \to \infty$, with $\rho^0 = \partial_x u^0$ while $\rho^k = \pi^k \partial_x u^k$ for $k = 1, 2$, where $u = (u^k)$ is the solution to* (32) *for a suitable initial condition (u_0^k) associated to ρ_0.*

The proof relies of several argument. As for the micro-macro derivation in the case of a 1:1 junction in Section 3.4, the key idea to determine the flux limiter is to count the number $N(t)$ of vehicles going through the junction between 0 and t. However this quantity is now random and, as far as we know, does not satisfy a superadditive property. To overcome this issue we show that its variance is controlled and we infer from this that its expectation is superadditive. This kind of argument was first developed in [3] in the framework of stochastic homogenization of HJ equations.

4.2 A model for the merging of two roads: approach by CL

Next we investigate the case of a 2:1 model, in which two entry roads merge into a single line. The model will depend on a priority rule, formalized, at the level of the conservation law, by a germ. A micro-macro derivation of such a rule is largely an open question (on this topic see [17]). For this reason, we follow here [19] and present a passage from a mesoscopic model to a macroscopic one. In the mesoscopic model the traffic flow is continuous, but the priority rule discrete: to fix the ideas, we discuss here the specific case of a time-periodic traffic light. In this setting, as only one entry line is active at each time, we are in the set-up of a 1:1 junction. After a time-space scaling, we end up with a macroscopic model, which is homogeneous in time. It involves a germ at the junction which keeps track of the traffic light.

Let us fix a few notation and assumptions, both valid in the meso and the macro models. Let $\mathcal{R}^0 = (0,\infty) \times \{0\}$ be the outgoing branch, $\mathcal{R}^j = (-\infty, 0) \times \{j\}$ for $j = 1, 2$ being the incoming ones. The set $\mathcal{R} = \bigcup_{j=0}^2 \mathcal{R}^j \cup \{0\}$ is endowed with the topology of three half lines glued together at the origin 0.

We make the following assumptions on the flux for some some $R > 0$.

$$\text{The flux } f : [0, R] \to [0,\infty) \text{ is of class } C^2, \text{ with } f'' < 0 \text{ on } [0, R],\ f(0) = f(R) = 0. \tag{33}$$

We set $f_{\max} := \max_{[0,R]} f$ and denote by f^+ the nondecreasing envelope of f and by f^- its nonincreasing envelope.

4.2.1 The mesoscopic model

We let I^1 (respectively I^2) denote the time sets on which the branch 1 (resp. the branch 2) is active in the mesoscopic model. The sets I^1 and I^2 form a partition of $\mathbb{R}$, each I^k, $k = 1, 2$, being periodic and of period 1 and locally the union of a finite number of intervals:

$$\begin{array}{c} I^1 \cup I^2 = \mathbb{R},\ I^1 \cap I^2 = \emptyset, \\ I^j \text{ is periodic of period 1 and consists locally in a finite number of intervals, } j = 1, 2. \end{array} \tag{34}$$

A typical example is the case where $I_1 = [0, \tau) + \mathbb{Z}$ (for some $\tau \in (0, 1)$) and $I_2 = \mathbb{R} \backslash I_1$. In this case the traffic light is green for road 1 on intervals of length τ and red on intervals of length $1 - \tau$.

Let ρ^j $(j = 0, 1, 2)$ be the density of vehicles. Then, in this mesoscopic model, $\rho = (\rho^0, \rho^1, \rho^2)$ solves

$$\begin{array}{llll} (i) & \rho^j \in [0, R] & \text{a.e. on} & (0,\infty)\times\mathcal{R}^j, \quad j = 0,1,2 \\ (ii) & \partial_t \tilde\rho^1 + \partial_x(f(\tilde\rho^1)) = 0 & \text{on} & I_1 \times \mathbb{R}, \qquad \tilde\rho^1 := \rho^1 \mathbf{1}_{(-\infty,0)} + \rho^0 \mathbf{1}_{(-\infty,0)} \\ & \partial_t \tilde\rho^2 + \partial_x(f(\tilde\rho^2)) = 0 & \text{on} & I_1 \times \mathbb{R} \qquad \tilde\rho^2 := \rho^2 \mathbf{1}_{(-\infty,0)} + R \mathbf{1}_{(-\infty,0)} \\ (iii) & \partial_t \tilde\rho^2 + \partial_x(f(\tilde\rho^2)) = 0 & \text{on} & I_2 \times \mathbb{R}, \qquad \tilde\rho^2 := \rho^2 \mathbf{1}_{(-\infty,0)} + \rho^0 \mathbf{1}_{(-\infty,0)} \\ & \partial_t \tilde\rho^1 + \partial_x(f(\tilde\rho^1)) = 0 & \text{on} & I_2 \times \mathbb{R} \qquad \tilde\rho^1 := \rho^1 \mathbf{1}_{(-\infty,0)} + R \mathbf{1}_{(-\infty,0)} \end{array} \tag{35}$$

In (ii), concerning the time intervals I_1, the first equation says that the traffic is unperturbed on the union $\mathcal{R}^1 \cup \mathcal{R}^2 \cup \{0\}$ for the density given by $\tilde\rho^1 := \rho^1 \mathbf{1}_{(-\infty,0)} + \rho^0 \mathbf{1}_{(-\infty,0)}$. On the other hand, the second equation says that the traffic is stopped on the entry road $\mathcal{R}^2$ at the level of the junction, because it is set to R and thus completely congested on the right of the junction. Equations (iii) describe the opposite configuration on the time intervals I_2. The existence and the uniqueness of a solution, given an initial condition (ρ_0^j), can be made by induction on each time interval. Note that the flow satisfies a Kato's inequality, because this is the case on each time interval.

In order to obtain a time-homogenous model, it is natural to scale the problem by looking at

$$\rho^\epsilon(t, x) = (\rho^{\epsilon,0}, \rho^{\epsilon,1}, \rho^{\epsilon,2})(t, x) := \rho(t/\epsilon, x/\epsilon),$$

where $\epsilon > 0$ is a small parameter and $\rho = (\rho^0, \rho^1, \rho^2)$ solves (35). The equation satisfied by the scaled density is almost the same as the one satisfied by the original one, except that the period of the traffic lights has been speeded up:

$$\begin{array}{llll} (i) & \rho^{\epsilon,j} \in [0, R] & \text{a.e. on} & (0,\infty)\times\mathcal{R}^j, \quad j = 0,1,2 \\ (ii) & \partial_t \tilde\rho^{\epsilon,1} + \partial_x(f(\tilde\rho^{\epsilon,1})) = 0 & \text{on} & \epsilon I_1 \times \mathbb{R}, \qquad \tilde\rho^{\epsilon,1} := \rho^{\epsilon,1} \mathbf{1}_{(-\infty,0)} + \rho^{\epsilon,0} \mathbf{1}_{(-\infty,0)} \\ & \partial_t \tilde\rho^{\epsilon,2} + \partial_x(f(\tilde\rho^{\epsilon,2})) = 0 & \text{on} & \epsilon I_1 \times \mathbb{R} \qquad \tilde\rho^{\epsilon,2} := \rho^{\epsilon,2} \mathbf{1}_{(-\infty,0)} + R \mathbf{1}_{(-\infty,0)} \\ (iii) & \partial_t \tilde\rho^{\epsilon,2} + \partial_x(f(\tilde\rho^{\epsilon,2})) = 0 & \text{on} & \epsilon I_2 \times \mathbb{R}, \qquad \tilde\rho^{\epsilon,2} := \rho^{\epsilon,2} \mathbf{1}_{(-\infty,0)} + \rho^{\epsilon,0} \mathbf{1}_{(-\infty,0)} \\ & \partial_t \tilde\rho^{\epsilon,1} + \partial_x(f(\tilde\rho^{\epsilon,1})) = 0 & \text{on} & \epsilon I_2 \times \mathbb{R} \qquad \tilde\rho^{\epsilon,1} := \rho^{\epsilon,1} \mathbf{1}_{(-\infty,0)} + R \mathbf{1}_{(-\infty,0)} \end{array} \tag{36}$$

Our aim is to understand the limit, as ϵ tends to 0, of ρ^ϵ. We will see that this limit is the solution of a scalar conservation law on $\mathcal{R}$, with a germ condition at the junction.

4.2.2 The macroscopic problem: an approach by germ

In order to describe the limit problem, we follow the ideas of [2, 29] already introduced in Subsection 3.2.2. We are interested in the solution $(\rho^k)_{k\in\{0,1,2\}}$ of a conservation law on the junction $\mathcal{R}$. The evolution depends on the flux f and on a germ $\mathcal{G} \subset [0, \infty)^3$:

$$\begin{array}{llll} (i) & \rho^j \in [0, R] & \text{a.e. on} & (0,\infty)\times\mathcal{R}^j, \quad j = 0,1,2 \\ (ii) & \partial_t \rho^j + \partial_x(f(\rho^j)) = 0 & \text{on} & (0,\infty)\times\mathcal{R}^j, \quad j = 0,1,2 \\ (iii) & (\rho^0(t, 0^-), \rho^1(t, 0^+), \rho^2(t, 0^+)) \in \mathcal{G} & \text{for a.e.} & t \in (0, \infty). \end{array} \tag{37}$$

Let us extend the notion of germs already introduced in Section 3.2.2. Recall that the pair (Kružkov entropy, entropy flux) is given, for $p, \bar p \in \mathbb{R}$, by

$$\eta(\bar p, p) = |p - \bar p|, \qquad q(\bar p, p) = \text{sign}(p - \bar p)(f(p) - f(\bar p)).$$

We define the box

$$Q := [0, R]^3 \tag{38}$$

and the subset of Q satisfying Rankine-Hugoniot condition

$$Q^{RH} := \{P = (p^0, p^1, p^2) \in Q, \quad f(p^0) = f(p^1) + f(p^2)\} \tag{39}$$

Let us introduce some terminology: For $P = (p^0, p^1, p^2)$, $\bar{P} = (\bar{p}^0, \bar{p}^1, \bar{p}^2) \in Q$, we define the **dissipation** by

$$D(\bar{P}, P) := \{q(\bar{p}^1, p^1) + q(\bar{p}^2, p^2)\} - q(\bar{p}^0, p^0) = \mathrm{IN} - \mathrm{OUT}$$

Consider a set $G \subset Q$. We say that G is a **germ** (for dissipation D) if

$$\begin{cases} G \subset Q^{RH} & \textbf{(Rankine-Hugoniot)} \\ D(\bar{P}, P) \geqslant 0 \quad \text{for all} \quad \bar{P}, P \in G & \textbf{(dissipation)} \end{cases}$$

We also say that G is **maximal** (for the dissipation D relatively to the box Q) if for every $P \in Q$, we have

$$\big(D(\bar{P}, P) \geqslant 0 \quad \text{for all} \quad \bar{P} \in G\big) \quad \Longrightarrow \quad P \in G.$$

Finally, we say that a set $E \subset \mathcal{G}$ **generates** $\mathcal{G}$, if, for any $U \in Q$,

$$\Big(D(U, \bar{U}) \geqslant 0 \qquad \forall \bar{U} \in E \Big) \qquad \Longrightarrow \qquad U \in \mathcal{G}.$$

By a solution to (37) we mean that $\rho \in L^\infty((0,\infty) \times \mathcal{R}, [0,R])$ is an entropy solution to the scalar conservation law in $(0,\infty) \times (\mathcal{R}\backslash\{0\})$ and that its trace $(\rho^0(t,0^+), \rho^1(t,0^-), \rho^2(t,0^-))$ at $x = 0$ belongs to $\mathcal{G}$ for a.e. $t \geqslant 0$. It is not difficult to check that $\rho \in L^\infty((0,\infty) \times \mathcal{R}, [0,R])$ is a solution to (37) if and only if it satisfies the following entropy inequality (for some set E generating $\mathcal{G}$):

$$\sum_{j=0}^{2} \left\{ \int_0^\infty \int_{\mathcal{R}^j} \eta(u^j, \rho^j)\partial_t \phi^j + q(u^j, \rho^j)\partial_x \phi^j + \int_{\mathcal{R}^j} \eta(u^j, \bar{\rho}^j)\phi^j(0,x) \right\} \geqslant 0 \tag{40}$$

for any $u = (u^j) \in E$ and any continuous nonnegative test function $\phi : [0,\infty) \times \mathcal{R} \to [0,\infty)$ with a compact support and such that $\phi^j := \phi_{|[0,+\infty)\times(\mathcal{R}^j \cup \{0\})}$ is C^1 for any $j = 0,1,2$. Note that (40) is just Kato's inequality between (ρ^j) and the constant solution (u^j).

It can be checked, exactly as in the proof of Theorem 3.5, that there is at most one solution to (37) given an initial condition and that the flow is an L^1-contraction. Existence of a solution is more subtle and requires additional condition on the germ.

4.2.3 A meso-macro derivation

Theorem 4.2 (Homogenization of the 1:2 junction [19])**.** *Under our standing assumptions, there exists a maximal germ $\mathcal{G} \subset Q$, such that the following holds true. Given an initial data $\bar{\rho}_0 = (\bar{\rho}_0^i) \in L^\infty(\mathcal{R})$ such that $\bar{\rho}_0^i \in [0,R]$ a.e. for $i = 0,1,2$, the solution ρ^ϵ of (36) with initial condition $\bar{\rho}_0$ converges in $L^1_{loc}([0,\infty) \times \mathcal{R})$ to the unique entropy solution ρ to*

$$\begin{array}{llll} (i) & \rho^j \in [0,R] & \textit{a.e. on} & (0,\infty) \times \mathcal{R}^j, \quad j = 0,1,2 \\ (ii) & \partial_t \rho^j + \partial_x (f(\rho^j)) = 0 & \textit{on} & (0,\infty) \times \mathcal{R}^j, \quad j = 0,1,2 \\ (iii) & (\rho^0(t,0), \rho^1(t,0), \rho^2(t,0)) \in \mathcal{G} & \textit{for a.e.} & t \in (0,\infty), \\ (iv) & \rho(0,\cdot) = \bar{\rho}_0 & \textit{on} & \{0\} \times \mathcal{R}, \end{array} \tag{41}$$

Remark 4.3. The micro-macro derivation remains an open question. In [17] we derived a germ model for a merging from a micro-model: it is however imperfect since the traffic light is supposed to be large when the number of vehicles is large, which is not physical. This assumption, which makes the micro model close to the mesoscopic regime, should be relaxed.

It is possible to build the set $\mathcal{G}$ explicitly from the set I_1: see [19]. For instance, when $I_1 = [0,\tau) + \mathbb{Z}$ (where $\tau \in (0,1)$), then, setting $\pi^1 = \tau$ and $\pi^2 = 1-\tau$,

$$\mathcal{G} := \left\{ P = (p^0,p^1,p^2) \in Q^{RH}, \quad \middle| \begin{array}{ll} 0 \leqslant f(p^k) \leqslant \pi^k \max f, & k=1,2 \\ \\ f^-(p^k) \geqslant \pi^k f^-(p^0), & k=1,2 \end{array} \right\}. \tag{42}$$

The first condition in (42) says that the maximal flux $f(p^1)$ coming from branch $\mathcal{R}^1$ at the junction is $\pi^1 f_{\max}$ (and symmetrically for the branch $\mathcal{R}^2$). The second condition expresses that—in a relaxed sense—the flux $f(p^0)$ at the right-hand side of the junction can be decomposed into a portion $f(p^1) = \pi^1 f(p^0)$ coming from branch $\mathcal{R}^1$ and $f(p^2) = \pi^2 f(p^0)$ coming from branch $\mathcal{R}^2$. The result of [19] is actually much more general and allows for a time-periodic flux limiter at the mesoscopic level.

The proof the theorem relies on the existence of suitable set $E \subset \mathcal{G}$ generating $\mathcal{G}$ and the construction of correctors. The notion of corrector is classical in homogenization. In our context, it reads:

Theorem 4.4 (Existence of correctors with prescribed values at infinity [19])**.** *For any $p = (p^0,p^1,p^2) \in E$, there exists an entropy solution $u_p = (u_p^i) \in L^\infty(\mathbb{R}\times\mathcal{R})$ of (35) which is 1-periodic in time and a constant $C>0$ such that for all $M \geqslant C$*

$$\|u_p^0 - p^0\|_{L^\infty(\mathbb{R}\times(-\infty,-M))} + \|u_p^i - p^i\|_{L^\infty(\mathbb{R}\times(M,\infty))} \leqslant CM^{-1}, \qquad i=1,2. \tag{43}$$

Sketch of proof of Theorem 4.2 from Theorem 4.4. Note that, if u_p is a corrector, then the scaled function $u_p^\epsilon(t,x) = u_p(t/\epsilon, x/\epsilon)$ solves (36) and strongly converges to $\sum_{k=0}^2 p^k \mathbf{1}_{\mathcal{R}^k}$. On the other hand, by Kato's inequality satisfied by the solutions of equation (36), we have

$$\sum_{j=0}^2 \left\{ \int_0^\infty \int_{\mathcal{R}^j} \eta(u^{\epsilon,j},\rho^{\epsilon,j})\partial_t\phi^j + q(u^{\epsilon,j},\rho^{\epsilon,j})\partial_x\phi^j + \int_{\mathcal{R}^j} \eta(u^{\epsilon,j},\rho_0^{\epsilon,j})\phi^j(0,x) \right\} \geqslant 0,$$

for any continuous nonnegative test function $\phi : [0,\infty)\times\mathcal{R} \to [0,\infty)$ with a compact support and such that $\phi^j := \phi_{|[0,+\infty)\times(\mathcal{R}^j\cup\{0\})}$ is C^1 for any $j=0,1,2$.

As the $\rho^{\epsilon,j}$ solve a scalar CL with a strictly convave flux function, they are bounded in BV far from the origin and thus converge, up to a subsequence, in L^1_{loc} to some functions ρ^j. So passing to the limit (up to this subsequence) as $\epsilon \to 0$ gives

$$\sum_{j=0}^2 \left\{ \int_0^\infty \int_{\mathcal{R}^j} \eta(p^j,\rho^j)\partial_t\phi^j + q(p^j,\rho^j)\partial_x\phi^j + \int_{\mathcal{R}^j} \eta(p^j,\rho_0^j)\phi^j(0,x) \right\} \geqslant 0.$$

This inequality, which holds for any $p \in E$, is exactly the entropy inequality (40) which characterizes the solution to (37). Thus any limit up to a subsequence of the compact sequence (ρ^ϵ) is equal to the solution ρ of (37), which shows that the whole family ρ^ϵ converges to ρ in L^1_{loc}. □

4.3 Open problems

There are many open problems in this area. I single out a few of them, mainly out of personal taste.

1. Perhaps the more important question is to derive a large class of admissible models of CL type including at the same time the approach by germs and the approach by flux limiters of HJ equations. Recall that the first one is L^1-conservative while the second one is $L^\infty-$contractive at the level of the anti-derivative; they are not equivalent in general. The difficulty is that the class of admissible conditions which have to be put at the junction is unclear. On see question, see the recent preprint [44].

2. In the line of this course, a micro-macro derivation of the continuous models should guide the building of these models.

3. One of the reasons it is interesting to have models of traffic flow is of course the design of junctions and the optimal control of the traffic: there are many works on the domain: see for instance the monograph [11] and the references therein. However I am not aware of a micro-macro derivation of these optimal control problems; it would also be interesting to explain how to use at a micro level the optimal strategies given at the continuous level.

4. Traffic flow is just one of the many instances of micro-macro derivation. Similar—but much more challenging—questions pop up in pedestrian flows for instance, with the main difference that the problem is no longer one dimensional and that, in this setting, one might take into account the fact that pedestrians anticipate the traffic. This question is closely related to mean field games...

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