A free boundary problem arising in mean field games

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The MFG problem

We are interested in the mean field game problem

$$(MFG) \qquad \begin{cases} -\partial_t u + \frac{1}{2}|\partial_x u|^2 = (m(t,x))^\theta & \text{in } (0,T)\times\mathbb{R}, \\ \partial_t m - \mathrm{div}(m\partial_x u) = 0 & \text{in } (0,T)\times\mathbb{R}, \\ m(0,x) = m_0(x) & \text{in } \mathbb{R}. \end{cases}$$

supplemented with either a terminal cost:

$$(MFG - terminal) \qquad u(T,x) = g(m(T,x))$$

or a constraint on $m(T)$:

$$(MFG - planning) \qquad m(T,x) = m_T(x)$$

where

- m_0 and m_T are compactly supported densities on $\mathbb{R}$,
- $\theta > 0$ and $g(r) = c_T r^\theta$.

System

$$(MFG - terminal) \qquad \begin{cases} -\partial_t u + \frac{1}{2}|\partial_x u|^2 = (m(t,x))^\theta & \text{in } (0,T)\times\mathbb{R}, \\ \partial_t m - \text{div}(m\partial_x u) = 0 \quad \text{in } (0,T)\times\mathbb{R}, \\ m(0,x) = m_0(x), \quad u(T,x) = g(m(T,x)) & \text{in } \mathbb{R}. \end{cases}$$

describes a game with infinitely many players in which

- a typical small player starting from x at time t minimizes the quantity

$$u(t,x) = \inf_{\gamma,\ \gamma(t)=x} \int_t^T \frac{1}{2}|\dot\gamma(s)| + (m(s,\gamma(s)))^\theta \, ds$$

- $m(t,\cdot)$ is the distribution of the players at time t when they play optimally.

(Lasry-Lions ('07), Huang-Caines-Malhamé ('07)).

Basic remarks

- Link with optimal transport: If (u, m) solves

$$(MFG - planning) \qquad \left\{ \begin{array}{l} -\partial_t u + \frac{1}{2}|\partial_x u|^2 = (m(t,x))^\theta \qquad \text{in } (0,T)\times\mathbb{R}, \\ \partial_t m - \text{div}(m\partial_x u) = 0 \qquad \text{in } (0,T)\times\mathbb{R}, \\ m(0,x) = m_0(x), \qquad m(T,x) = m(T,x) \qquad \text{in } \mathbb{R}. \end{array} \right.$$

then $(m, -\partial_x u)$ is a minimizer of the optimal transport problem

$$\inf_{(m,\ \alpha)} \int_0^T \int_R \frac{1}{2}|\alpha|^2 + \frac{m^{\theta+1}}{\theta+1}$$

where the infimum is taken over (m, α) such that

$$\partial_t m + \mathrm{div}(m\alpha) = 0, \qquad m(0) = m_0,\ m(T) = m_T.$$

- Hamiltonian system: (MFG) corresponds formally to the Hamiltonian system associated with the Hamiltonian

$$\mathcal{H}(u,m) = \int_0^T \int_{\mathbb{R}} \frac{1}{2} m(\partial_x u)^2 - \frac{m^{\theta+1}}{\theta+1}$$

A few references

From the MFG system:

$$(MFG) \qquad \begin{cases} -\partial_t u + \frac{1}{2}|\partial_x u|^2 = m^\theta \\ \partial_t m - \operatorname{div}(m\partial_x u) = 0. \end{cases}$$

- Lions' a priori estimates:
  - $m \in L^\infty$, $u \in W^{1,\infty}$,
  - (u, m) smooth in $\{m > 0\}$.

- Weak formulation: Existence/uniqueness of a weak solution by C. ('15), C.-Graber ('15), C.-Graber-Porretta-Tonon ('15), Munoz ('22).

- Regularization $L^1 \to L^\infty$/displacement convexity: Lavenant-Santambrogio ('18), Gomes-Seneci ('18), Porretta ('23).

- Classical solutions: with periodic boundary conditions,
  - for entropic coupling: Munoz ('22), Porretta ('23),
  - for positive initial densities: Mimikos-Munoz ('23),

$\longrightarrow$ **Main novelty here:** problems in which $\{m > 0\}$ is not the whole space.

Outline

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Self-similar solution

Recall the MFG system:

$$(MFG) \qquad \begin{cases} -\partial_t u + \frac{1}{2}|\partial_x u|^2 = m^\theta \\ \partial_t m - \mathrm{div}(m\partial_x u) = 0. \end{cases}$$

Proposition

There exists a self-similar solution to (MFG) given by

$$m(t,x) = t^{-\alpha}\phi(t^{-\alpha}x), \text{ where } \qquad \phi(x) = \left(\frac{\alpha(1-\alpha)}{2}\right)^{1/\theta}\left(R^2 - x^2\right)^{1/\theta}, \ \alpha = \frac{2}{2+\theta},$$

$$u(t,x) = -\alpha\frac{x^2}{2t} - Ct^{2\alpha-1} \qquad \text{in } \{m > 0\} \qquad (\text{for } \theta \neq 2),$$

where $C = R^2\frac{\alpha(1-\alpha)}{2\alpha-1}$ and R is such that $\int_{\mathbb{R}} m(t,x)dx = 1$.

Remarks on the self-similar solution

Recall that $m(t,x) = t^{-\alpha}\phi(t^{-\alpha}x)$ and $u(t,x) = -\alpha\frac{x^2}{2t} - Ct^{2\alpha-1}$ is a a self similar solution where

$$\phi(x) = \left(\frac{\alpha(1-\alpha)}{2}\right)^{1/\theta}\left(R^2 - x^2\right)^{1/\theta}, \ \alpha = \frac{2}{2+\theta} \in (0,1),$$

Remarks:

- $m(t,\cdot)$ has a compact support in $[-Rt^{\alpha}, Rt^{\alpha}]$, is Holder continuous (but not globally smooth). Moreover m^{θ} is Lipschitz.
- (behavior at $t=0$) $\lim_{t\to 0^+} m(t,\cdot) = \delta_0$.
- Optimal trajectories solve $\dot{\gamma} = -\partial_x(t,\gamma(t)) = \alpha\gamma(t)/t$. Hence $\gamma(t) = ct^{\alpha}$, $t \geq 0$, $c \in [-R,R]$.
- u can be extended into a C^1 function in the whole space (but *not* C^2).

Outline

Assumptions

- m_0 and m_T are smooth in their support,
- with $\{m_0 > 0\} = (a_0, b_0)$ and $\{m_T > 0\} = (a_T, b_T)$,
- (Compatibility) For some $\alpha_0 > 0$,

$$\frac{1}{C_0}\text{dist}(x, \{a_0, b_0\})^{\alpha_0} \leq m_0(x) \leq C_0\text{dist}(x, \{a_0, b_0\})^{\alpha_0}$$

and

$$\frac{1}{C_T}\text{dist}(x, \{a_T, b_T\})^{\alpha_0} \leq m_T(x) \leq C_T\text{dist}(x, \{a_T, b_T\})^{\alpha_0}.$$

- For $(MFG - terminal)$, the terminal cost is $g(r) = c_T r^\theta$.

Main results

Theorem

Let (u, m) be the solution to (MFG-terminal), or to (MFG-planning).

- (Regularity) There exists $\gamma > 0$ such that

$$m \in C^{\gamma}([0,T]\times\mathbb{R}) \cap C^{1,\beta}_{loc}(\{m>0\}), \qquad u \in C^{1+\gamma/2}([0,T]\times\mathbb{R}) \cap C^{2,\beta}_{loc}(\{m>0\})$$

- (Bounded support) Moreover there exist two functions $\gamma_L < \gamma_R \in W^{1,\infty}(0,T)$, such that

$$\{m>0\} = \{(x,t) \in \mathbb{R}\times[0,T] : \gamma_L(t) < x < \gamma_R(t)\}.$$

- (Convexity of the support) If we assume further the concavity condition

$$(m_0^{\theta})_{xx} \le 0 \text{ in } \{x \in (a_0,b_0) : dist(x,\{a_0,b_0\}) < \delta\} \text{ for some } \delta > 0,$$

  then $\gamma_L, \gamma_R \in W^{2,\infty}(0,T)$, and there exists $K>0$ such that, for a.e. $t \in [0,T]$,

$$\frac{1}{K} \le \ddot{\gamma}_L(t) \le K, \text{ and } -K \le \ddot{\gamma}_R(t) \le -\frac{1}{K}.$$

Sketch of proof (1): basic a priori estimates

- (Lions' key remark) Let (u, m) be a classical solution to (MFG). The map u satisfies the **quasilinear elliptic equation**

$$-u_{tt} + 2u_x u_{xt} - (u_x^2 + mf'(m))u_{xx} = 0 \qquad \text{in } \mathbb{R} \times (0, T)$$

with $f(m) = m^\theta$, $m = f^{-1}(-u_t + u_x^2/2)$. This yields

$$\|m\|_\infty < \infty, \qquad \|\partial_x u\|_\infty < \infty.$$

- (Displacement convexity) If in addition $h : (0, \infty) \to \mathbb{R}$ is twice differentiable, then

$$\frac{d^2}{dt^2} \int_{\mathbb{T}} h(m) = \int_{\mathbb{T}} mh''(m)(mu_{xx}^2 + f'(m)m_x^2).$$

(Gomes-Seneci ('18), Mimikos-Munoz ('23), Porretta ('23))

- (Continuity of u) The map $v = f(m)$ satisfies an elliptic equation with no zero–order terms, with $|D_{(t,x)}v|$ belongs to L^2_{loc}. Hence $v = f(m)$ has a modulus of continuity (Lebesgue argument in dim. 2).

Sketch of proof (2): characteristic flow

- We consider the flow of optimal trajectories

$$\begin{cases} \dot\gamma(t,x) = -\partial_x u(t,\gamma(t,x)) & t \geq 0 \\ \gamma(0,x) = x \end{cases}$$

where $x \in (a_0, b_0)$ is in the support of m_0.

- Note that $m(t) = \gamma(t,\cdot)\sharp m_0$.

- We derive from this the mass preservation equality:

$$\gamma_x(t,x) = \frac{m_0(x)}{m(t,\gamma(t,x))}.$$

- **Key remark:** Taking the derivative in t and using the equation for u yields that γ solves in $(a_0, b_0) \times (0, T)$ the elliptic equation

$$\gamma_{tt} + \frac{\theta m_0^\theta}{(\gamma_x)^{2+\theta}} \gamma_{xx} = \frac{(m_0^\theta)_x}{(\gamma_x)^{1+\theta}}.$$

Sketch of proof (3): characteristic flow (continued)

- Recall the **key remark:** γ is a solution in $(a_0, b_0) \times (0, T)$ to the elliptic equation

$$\gamma_{tt} + \frac{\theta m_0^\theta}{(\gamma_x)^{2+\theta}} \gamma_{xx} = \frac{(m_0^\theta)_x}{(\gamma_x)^{1+\theta}}.$$

- By barrier argument, we get the bounds

$$C^{-1} \le \gamma_x(t,x) \le C.$$

- From uniform elliptic regularity we deduce that

$$\gamma \in W^{1,\infty}((a_0, b_0) \times (0, T)) \cap C^{2,\alpha}_{loc}((a_0, b_0) \times [0, T]),$$

with $\gamma_L(t) = \gamma(a_0, t)$, $\gamma_R(t) = \gamma(b_0, t)$.

- As $m(t, \cdot) = \gamma(t, \cdot)\sharp m_0$, we infer the interior regularity of m.

Sketch of proof (4): regularity of (u, m) in the whole space

- The C^β regularity of m
  comes from an intrinsic scaling argument à la Di Benedetto on $v(t,x) = f(m(t,\gamma(t,x))$
  (Harnack inequality, intrinsic Caccioppoli inequality, De Giorgi Lemma, reduction of oscillation in intrinsic rectangles)

- $C^{1,\beta/2}$ regularity of u.
  - the maps u and $-u(T-t,\cdot)$ satisfy a HJ eq. with convex Hamiltonian and Holder RHS,
  - yields to semi-concavity of u and $-u$ (with a nonlinear modulus)
  - and thus to a $C^{1,\beta/2}$ regularity.
  (as in Cannarsa-Soner ('89))

Outline

The MFG-planning problem

We are now interested in the mean field game problem with a singular initial condition

$$(MFG-planning) \qquad \left\{ \begin{array}{ll} -\partial_t u + \frac{1}{2}|\partial_x u|^2 = (m(t,x))^\theta & \text{in } (0,T)\times\mathbb{R}, \\ \partial_t m - \text{div}(m\partial_x u) = 0 & \text{in } (0,T)\times\mathbb{R}, \\ m(0,dx) = \delta_0(dx), \quad m(T,x) = m_T(x) & \text{in } \mathbb{R}. \end{array} \right.$$

where m_T is a smooth density, supported in $[a_T, b_T]$ and satisfying the compatibility condition

$$C^{-1}\,\mathrm{dist}(x,\{a_T,b_T\})^{1/\theta} \leq m_T(x) \leq C\,\mathrm{dist}(x,\{a_T,b_T\})^{1/\theta}.$$

Back to the self-similar solution

$$(MFG) \qquad \begin{cases} -\partial_t u + \frac{1}{2}|\partial_x u|^2 = m^\theta \\ \partial_t m - \mathrm{div}(m \partial_x u) = 0. \end{cases}$$

Proposition

There exists a self-similar solution to (MFG) given by

$$m(t,x) = t^{-\alpha}\phi(t^{-\alpha}x), \text{ where } \qquad \phi(x) = \left(\frac{\alpha(1-\alpha)}{2}\right)^{1/\theta}\left(R^2 - x^2\right)^{1/\theta}, \; \alpha = \frac{2}{2+\theta},$$

$$u(t,x) = -\alpha\frac{x^2}{2t} - R^2\frac{\alpha(1-\alpha)}{2\alpha - 1}t^{2\alpha-1} \qquad \text{in } \{m > 0\} \qquad (\text{for } \theta \neq 2),$$

where R is such that $\int_{\mathbb{R}} m(t,x)dx = 1$. It satisfies

$$\lim_{t\to 0^+} m(t,\cdot) = \delta_0.$$

Remark: Note that $u(t,\cdot)$ blows up if $2\alpha - 1 < 0 \Leftrightarrow \theta > 2$.

Main result

$$(MFG-planning)\quad \begin{cases} -\partial_t u + \frac{1}{2}|\partial_x u|^2 = (m(t,x))^\theta & \text{in } (0,T)\times\mathbb{R}, \\ \partial_t m - \operatorname{div}(m\partial_x u) = 0 & \text{in } (0,T)\times\mathbb{R}, \\ m(0,dx) = \delta_0(dx), \quad m(T,x) = m_T(x) & \text{in } \mathbb{R}. \end{cases}$$

Theorem

- (Existence) There exists a solution (u, m) to $(MFG - planning)$ with u continuous in $(0, T)\times\mathbb{R}$ and $m \in L^\infty_{loc}((0, T]\times\mathbb{R}) \cap C^0([0, T], \mathcal{P}_2)$, (u, m) smooth in $\{m > 0\}$,
- (Behavior at $t = 0$) which in addition is such that $t^\alpha m(t, t^\alpha\cdot)$ converges locally uniformly in $(-R, R)$ to the self-similar profile ϕ as $t \to 0^+$.
- (Uniquness) If $\theta \in (0, 2)$, the solution to $(MFG - planning)$ is unique.

Remarks:

- Munoz ('24) studies the convergence to the self similar solution as $T \to \infty$.
- When $m_0 = m_T = \delta_0$ and $\theta = 1$, Lions-Souganidis ('24) gives an explicit formula for the unique solution and show that it is the limit of a viscous approximation.

Construction of the solution

- Let $(u^\varepsilon, m^\varepsilon)$ be the solution to

$$\begin{cases} -\partial_t u^\varepsilon + \frac{1}{2}|\partial_x u^\varepsilon|^2 = (m^\varepsilon(t,x))^\theta & \text{in } (0,T)\times\mathbb{R}, \\ \partial_t m^\varepsilon - \text{div}(m^\varepsilon \partial_x u^\varepsilon) = 0 & \text{in } (0,T)\times\mathbb{R}, \\ u^\varepsilon(0,x) = m_0^\varepsilon(x) := \varepsilon^{-\alpha}\phi(\varepsilon^{-\alpha}x), \quad m^\varepsilon(T,x) = m_T(x) & \text{in } \mathbb{R}. \end{cases}$$

By the previous analysis we know that $(u^\varepsilon, m^\varepsilon)$ exists, is unique and is smooth in $\{m^\varepsilon > 0\}$.

- Let $\gamma^\varepsilon = \gamma^\varepsilon(t,x)$ be the flow of optimal solutions, for $x \in \text{Spt}(m_0^\varepsilon) = [-R\varepsilon^\alpha, R\varepsilon^\alpha]$. Set

$$\tilde{\gamma}^\varepsilon(t,y) = \gamma^\varepsilon(t, \varepsilon^\alpha y) \qquad y \in [-R,R].$$

Then $\text{m}^\varepsilon(t) = \tilde{\gamma}^\varepsilon(t,\cdot)\sharp\phi$ and $\tilde{\gamma}^\varepsilon$ solves

$$\tilde{\gamma}^\varepsilon_{tt} + \frac{\theta\phi^\theta}{(\tilde{\gamma}^\varepsilon_y)^{2+\theta}}\tilde{\gamma}^\varepsilon_{yy} = \frac{(\phi^\theta)_y}{(\tilde{\gamma}^\varepsilon_y)^{\theta+1}} \qquad \text{in } (0,T)\times(-R,R).$$

Construction of the solution (continued)

Lemma

$$C^{-1} \leq \tilde{\gamma}^{\varepsilon}_y \leq C, \qquad \text{for some } C > 1 \text{ independent of } \varepsilon.$$

Sketch of proof of the lemma

- (Friendly giant) There exists k large such that, for any $\delta > 0$ small

$$w(t,y) = k(t + \varepsilon + \delta\phi^{-(2\theta+1)}(y))^{\alpha}$$

  is a super-solution to $(*)$. This gives the upper-bound as $\delta \to 0$.
- (bound below) Barrier argument on $v^{\varepsilon}(t,y) = (m^{\varepsilon})^{\theta}(t, \tilde{\gamma}^{\varepsilon}(t,y)) = \frac{\phi^{\theta}(y)}{(\tilde{\gamma}^{\varepsilon}_y(t,y))^{\theta}}$.

Consequences:

- Uniform bounds for $\tilde{\gamma}^{\varepsilon}$: $|\tilde{\gamma}^{\varepsilon}(t,y)| \leq C(t+\varepsilon)^{\alpha}$,
- As $\tilde{\gamma}^{\varepsilon}$ solves the elliptic equation

$$(*) \qquad \tilde{\gamma}^{\varepsilon}_{tt} + \frac{\theta\phi^{\theta}}{(\tilde{\gamma}^{\varepsilon}_y)^{2+\theta}}\tilde{\gamma}^{\varepsilon}_{yy} = \frac{(\phi^{\theta})_y}{(\tilde{\gamma}^{\varepsilon}_y)^{\theta+1}} \qquad \text{in } (0,T)\times(-R,R).$$

  we obtain the uniform smoothness of $\tilde{\gamma}^{\varepsilon}$,
- Uniform bounds: $\|m^{\varepsilon}(t)\|_{\infty} \leq C(t+\varepsilon)^{-\alpha}$ and $\int_{\mathbb{R}} x^2 m^{\varepsilon}(t,x)dx \leq C(t+\varepsilon)^{2\alpha}$.
- $\longrightarrow$ Existence of a solution to (MFG-planning): $(u, m, \tilde{\gamma}) = \lim_{\varepsilon\to 0}(u^{\varepsilon}, m^{\varepsilon}, \tilde{\gamma}^{\varepsilon})$.

Construction of the solution: behavior at $t = 0$

- Following Munoz ('24), we change variables: for $t = e^\tau,\ x = t^\alpha \eta$, let

$$\mu(\tau,\eta) = t^\alpha m(t,x),\ w(\tau,\eta) = t^{1-2\alpha}u(t,x) + \frac{\alpha}{2}\eta^2,\ \hat\gamma(\tau,y) = t^\alpha \tilde\gamma(t,y).$$

- Equations for $(w,\mu,\hat\gamma)$: (w,μ) solves the MFG system

$$\begin{cases} -w_\tau + \frac{1}{2}|w_\eta|^2 = \mu^\theta + \frac{\alpha(1-\alpha)}{2}\eta^2 + (2\alpha-1)w & \text{in } (-\infty,\ln(T))\times\mathbb{R} \\ \mu_\tau - (\mu w_\eta)_\eta = 0 \end{cases}$$

while $\hat\gamma$ solves the elliptic equation

$$\alpha(\alpha-1)\hat\gamma + (2\alpha-1)\hat\gamma_\tau + \hat\gamma_{\tau\tau} + \frac{\theta\phi^\theta}{(\hat\gamma_y)^{2+\theta}}\hat\gamma_{yy} = \frac{(\phi^\theta)_y}{(\hat\gamma_y)^{\theta+1}} \text{ in } (-\infty,\ln(T))\times(-R_\alpha,R_\alpha).$$

Note that $\mu(\tau) = \hat\gamma(t,\cdot)\sharp\phi$.

- Energy estimates yield $\lim_{\tau\to-\infty}\hat\gamma(\tau,y) = y$, which implies $\lim_{\tau\to-\infty}\mu(\tau) = \phi$.

Conclusion and open problems

Summary

- We have proved existence and regularity of the solution to the MFG system

$$(MFG - planning) \qquad \begin{cases} -\partial_t u + \frac{1}{2}|\partial_x u|^2 = (m(t,x))^\theta & \text{in } (0,T)\times\mathbb{R}, \\ \partial_t m - \text{div}(m\partial_x u) = 0 & \text{in } (0,T)\times\mathbb{R}, \\ m(0,dx) = m_0(dx), \quad m(T,x) = m_T(x) & \text{in } \mathbb{R}. \end{cases}$$

  when m_0 has a compact support.
- Link with the self-similar solution when m_0 is a Dirac mass.

Open problems

- **For general initial conditions:**
  - Smoothness of the free boundary
  - multi-dimensional case
- **For the singular initial condition:** uniqueness of the solution when $\theta \geq 2$.

Thank you!

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