Fully nonlinear equations in thin domains: a test function approach

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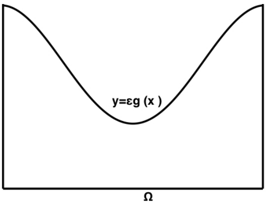
Rencontres Normandes sur les aspects théoriques et numériques des EDP

The classical result of Hale and Raugel in thin domains studies the asymptotic behaviour, as ε goes to zero, of u_ε, solution of

$$\begin{cases} -\Delta u_\varepsilon + u_\varepsilon = f(x,y) & \text{in } \Omega_\varepsilon \\ \partial_{\nu_\varepsilon} u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \end{cases}$$

where $\Omega_\varepsilon = \{(x,y) \in \mathbb{R}^N \times \mathbb{R} : x \in \Omega,\ 0 < y < \varepsilon g(x)\}$, for some



g such that $0 < \inf_\Omega g \le \sup_\Omega g < \infty$.

Hale and Raugel (1989) have proved:

H-R

If $g \in C^3(\overline{\Omega})$ then u_ε converges to u_o solution of

$$\begin{cases} -(\Delta u_o + \frac{Dg \cdot Du_o}{g}) + u_o = f(x,0) & \text{in } \Omega \\ \partial_\nu u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

This result has been extended in a wide variety of related problems all in a variational setting:

- **Arrieta, Pereira** *Homogenization in a thin domain with an oscillatory boundary* J. Math. Pure Appl (2011)
- **Arrieta, Nogueira, Pereira**.... *oscillatory boundaries* Comput. Math. Appl. (2019)
- **Arrieta, Villanueva-Pesqueira**..... *doubly weak oscillatory boundary*, Preprint 2024
- **Arrieta, Nakasato, Villanueva-Pesqueira** *Homogenization in 3D thin domains* .... Preprint 2024

$$-\Delta u_\varepsilon + u_\varepsilon = f(x,y) \to -(\Delta u_o + \frac{Dg \cdot Du_o}{g}) + u_o = f(x,0)$$

Where does the "new term" come from? If we make the change of variables $X = x$, and $Y = y\varepsilon g(x)$ i.e. Ω_ε becomes the flat cylindrical set $Q = \Omega \times (0,1)$ then the equation becomes

$$\begin{cases} -\frac{1}{g}\mathrm{div}(\mathrm{B}_\varepsilon \mathrm{D}\mathrm{v}_\varepsilon) + \mathrm{v}_\varepsilon = \mathrm{f} & \text{in } Q \\ \partial_\nu v_\varepsilon = 0 & \text{on } \partial Q \end{cases}$$

with

$$B_\varepsilon D v_\varepsilon = \begin{bmatrix} gDv - yv_y Dg \\ -yDg \cdot Dv + \frac{v_y}{\varepsilon^2 g}(1 + \varepsilon^2 |Dg|^2 y) \end{bmatrix}.$$

So using the weak solution formulation:

$$\int_Q (\frac{1}{g} B_\varepsilon Dv_\varepsilon \cdot D\varphi - B_\varepsilon \frac{Dv_\varepsilon \cdot Dg}{g^2}\varphi) dxdy + \int_Q v_\varepsilon \varphi dxdy = \int_Q f(x, \varepsilon g(x) y)\varphi dxdy$$

By a priori bounds, $\partial_y v_\varepsilon \to 0$ and hence the limit equation becomes

$$\int_\Omega (Dv_o \cdot D\varphi - \frac{Dv_o \cdot Dg}{g}\varphi) dx + \int_\Omega v_o \varphi dx = \int_\Omega f(x, 0)\varphi dx.$$

But if the equation is not variational, what happens? how do we recover the limit equation. Use L. C. Evans test function approach.

Heuristic

Where does the term come from without integration by part?
Recall that for $v_\varepsilon(x,y) := u_\varepsilon(x, \varepsilon g(x) y)$:

$$-\mathrm{tr}(D^2 u_\varepsilon) + u_\varepsilon = f \text{ becomes } -\mathrm{tr}(\tilde{B}_\varepsilon D^2 v_\varepsilon) + v_\varepsilon = f.$$

Using standard a priori estimate, we can suppose that

$$\partial_{yy} v_\varepsilon \leq C\varepsilon^2.$$

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So we may use the following *ansatz*:

$$v_\varepsilon(x,y) = w(x) + \varepsilon^2 k(x) \frac{y^2}{2} + o(\varepsilon^2).$$

Substituting the ansatz in the equation, we let, formally ε go to zero, after a tedious but simple computation it is easy to see that we obtain

$$-\mathrm{tr}\left(\begin{pmatrix} D^2 w(x) & 0 \\ 0 & \frac{k(x)}{g^2(x)} \end{pmatrix}\right) + w = f.$$

We still don't know what is $k(x)$! Using the ansatz, in the Neumann condition on the "top" boundary, gives

$$Dg(x) \cdot [Dw(x) + \varepsilon^2 Dk(x)\frac{1}{2}] = \frac{k(x)}{g(x)}[1 + \varepsilon^2 Dg(x)]$$

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Passing to the limit we find

$$k(x) = g(x)Dg(x)\cdot Dw(x) \text{ i.e. } \frac{k(x)}{g^2} = \frac{Dg(x)\cdot Dw(x)}{g}$$

i.e. the limit equation is indeed.

$$\begin{cases} -\mathrm{tr}\left(\begin{pmatrix} D^2 w & 0 \\ 0 & Dg\cdot Dw/g \end{pmatrix}\right) + w = f(x,0) \ \text{ in } \Omega, \\ \frac{\partial w}{\partial \nu} = 0 \ \text{ on } \partial\Omega. \end{cases}$$

Oblique boundary condition-BBI. The domain Ω_ε

In the work I am presenting we consider, a larger class of domains, more general boundary conditions and a much wider class of operators.

Oblique boundary condition-BBI. The domain Ω_ε

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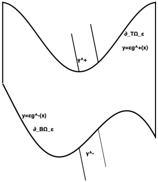
For ε small, $\Omega_\varepsilon = \{(x, y) \in \Omega \times \mathbb{R} : \varepsilon g^-(x) < y < \varepsilon g^+(x)\}$, $\Omega \subset \mathbb{R}^N$. We denote by

- $\partial_T \Omega_\varepsilon$ the top boundary i.e. $y = \varepsilon g^+(x)$,
- $\partial_B \Omega_\varepsilon$ the bottom of the domain i.e. $y = \varepsilon g^-$
- $\partial_L \Omega_\varepsilon$ the lateral boundary i.e. $\partial\Omega \times [\varepsilon g^-, \varepsilon g^+]$.



We suppose that $g^\pm \in C^1(\overline{\Omega})$ and $g^- < g^+$.

Because of the corners, in the framework of viscosity solutions, we will need to treat with special care the boundary.

Oblique boundary condition-BBI. The operator.

We treat fully nonlinear equations in thin domains i.e.

$$F(D^2u, Du, u, (x,y)) = 0 \text{ in } \Omega_\varepsilon$$

where $F : \mathcal{S}(N+1) \times \mathbb{R}^{N+1} \times \mathbb{R} \times \Omega_\varepsilon \to \mathbb{R}$ is a proper functional in the sense of the User's guide

$$(H2) \begin{cases} F \in C(\mathcal{S}(N+1) \times \mathbb{R}^{N+1} \times \mathbb{R} \times \Omega_\varepsilon, \mathbb{R}), \\ F(X, p, r, (x,y)) \leq F(Y, p, s, (x,y)) \text{ for } r \leq s, \text{ and } Y \leq X. \end{cases}$$

Furthermore, we strengthen the monotonicity condition on F in the above as follows.

(H3) There exists $\alpha > 0$ such that

$$\alpha(r-s) \leq F(X, p, r, (x,y)) - F(X, p, s, (x,y))$$

for $r \geq s$ and $(X, p, (x,y)) \in \mathcal{S}(N+1) \times R^{N+1} \times \Omega_\varepsilon$.

We shall call these operators "proper".

The boundary condition $\Omega_\varepsilon = \{(x,y) \in \Omega \times \mathbb{R} : \varepsilon g^-(x) < y < \varepsilon g^+(x)\}$

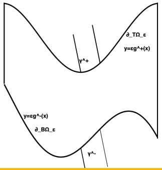
In general we can treat operators that are degenerate elliptic or non linear, but for clarity sake, in this talk, I will begin by presenting the results for $F(D^2u, u) = -\Delta u + u$ i.e. we consider problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega_\varepsilon \\ \gamma \cdot Du = \beta & \text{on } \partial\Omega_\varepsilon \end{cases}$$

Where the boundary condition will be meant in the viscosity sense and will be written:

$$\gamma^+ \cdot Du^\varepsilon = \beta^+ \ \text{ on } \partial_T\Omega_\varepsilon \quad \text{and} \quad \gamma^- \cdot Du^\varepsilon = \beta^- \ \text{ on } \partial_B\Omega_\varepsilon,$$

$$\gamma_L \cdot Du^\varepsilon = \beta_L \ \text{ on } \partial_L\Omega_\varepsilon.$$



Oblique boundary condition

Concerning conditions

$$\gamma^+ \cdot Du^\varepsilon = \beta^+ \text{ on } \partial_T\Omega_\varepsilon \quad \text{and} \quad \gamma^- \cdot Du^\varepsilon = \beta^- \text{ on } \partial_B\Omega_\varepsilon,$$

we require that $\gamma^\pm \cdot \nu_\varepsilon > 0$. Hence if we write

$$\gamma^\pm = (\gamma_1^\pm, \gamma_2^\pm) \text{ with } \gamma_1(x,y)^\pm \in \mathbb{R}^N \text{ and } \gamma_2(x,y)^\pm \in \mathbb{R}$$

Then without loss of generality, since $\pm\gamma_2^\pm > 0$, we can choose $\gamma_2^\pm = \pm 1$.

$$\gamma_1^+ \cdot D_x u^\varepsilon + u_y^\varepsilon = \beta^+ \text{ on } \partial_T\Omega_\varepsilon \quad \text{and} \quad \gamma_1^- \cdot D_x u^\varepsilon - u_y^\varepsilon = \beta^- \text{ on } \partial_B\Omega_\varepsilon,$$

We assume that for some $k^{\pm} \in C(\overline{\Omega}, \mathbb{R}^N)$ and $l^{\pm} \in C(\overline{\Omega}, \mathbb{R})$,

$$\begin{cases} \gamma_1^{\pm}(x,y) = \gamma_1^{\pm}(x,0) + k^{\pm}(x)y + o(|y|) \\ \beta^{\pm}(x,y) = \beta^{\pm}(x,0) + l^{\pm}(x)y + o(|y|), \end{cases}$$

as $y \to 0$, where $o(|y|)/|y| \to 0$ uniformly on $\overline{\Omega}$ as $y \to 0$.
A crucial assumption on $\gamma^{\pm}$ and $\beta^{\pm}$ for $x \in \overline{\Omega}$

$\beta^+(x,0) = -\beta^-(x,0) := \beta_o(x)$ and $\gamma_1^+(x,0) = -\gamma_1^-(x,0) := \gamma_o(x)$.

Under the above conditions on the boundary

Theorem (I.B., A. Briani, H. Ishii)

Let u^ε be a viscosity solution to $-\Delta u^\varepsilon + u^\varepsilon = f$ in Ω_ε with boundary conditions $\gamma \cdot Du^\varepsilon = \beta$ as in (17) for $\varepsilon \in (0, \varepsilon_0]$. Then, u^ε converges to u^0 a solution in Ω of

$$-\Delta u - D^2 u\, \gamma_o(x) \cdot \gamma_o(x) - b(x) \cdot Du - c(x) u + u = f(x, 0).$$

Satisfying $\gamma_L(x, 0) \cdot u = \beta_L(x, 0)$ on $\partial\Omega$. Furthermore

$$\lim_{\varepsilon \to 0^+} \max_{(x,y) \in \overline{\Omega_\varepsilon}} |u^\varepsilon(x, y) - u^0(x)| = 0.$$

The values of $b(x)$ and $c(x)$ will be written in the next slide.
For general operators F, for which we require only to be proper, the convergence may not be uniform.

Precisely

$$b(x) = \gamma_o (D\gamma_o)^T - \frac{1}{g^+ - g^-}\Big(g^+ k^+ + g^- k^-\Big), \qquad (1)$$

$$c(x) = -\gamma_o \cdot D\beta_o + \frac{1}{g^+ - g^-}\Big(g^+ l^+ + g^- l^-\Big). \qquad (2)$$

Recall that $\gamma^\pm = (\gamma_1^\pm, \pm 1)$,

$$\begin{cases} \gamma_1^\pm(x,y) = \pm\gamma_o(x) + k^\pm(x)y + o(|y|) \\ \beta^\pm(x,y) = \pm\beta_o(x) + l^\pm(x)y + o(|y|), \end{cases}$$

In the general case, if we consider the problem (P_ε)

$$F(D^2u_\varepsilon, Du_\varepsilon, u_\varepsilon, (x,y)) = 0 \text{ in } \Omega_\varepsilon,\ \gamma\cdot Du_\varepsilon = \beta \text{ on } \partial\Omega_\varepsilon,$$

and we call S_ε the set of viscosity solutions of this problem.
The limit equation will be defined through
$G : \mathcal{S}(N)\times \mathbb{R}^N \times \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ by

$$G(D^2u, Du, u, x) = F(A+B+C, (Du, \beta_o - \gamma_o\cdot Du), u, (x,0)),$$

where

$$\begin{aligned} A &= \begin{pmatrix} D^2u & -D^2u\gamma_o(x)^T \\ -\gamma_o(x)D^2u & \gamma_o(x)D^2u\ \gamma_o(x)^T \end{pmatrix}, \\ B &= \begin{pmatrix} 0 & -(DuD\gamma_o(x))^T \\ -DuD\gamma_o(x) & b(x)\cdot Du \end{pmatrix}, \\ C &= \begin{pmatrix} 0 & D\beta_o(x)^T \\ D\beta_o(x) & c(x) \end{pmatrix}. \end{aligned}$$

So we call (P_o), the problem

$$G(D^2u, Du, u, x) = 0 \text{ in } \Omega,\ \gamma_L\cdot Du = \beta_L(x,0) \text{ on } \partial\Omega.$$

Theorem (I.B., A. Briani, H. Ishii)

Assume that F is proper, that Ω_ε, $g^{\pm}$, γ and β satisfy the conditions described above. Let S_ε be the set of viscosity solutions to the problem (P_ε) If we define the half relaxed limits $u^{\pm}$ by

$$u^+(x) = \lim_{r\to 0^+} \sup_{u\in S_\varepsilon} \{u(\xi,\eta) : (\xi,\eta)\in \overline{\Omega_\varepsilon},\ 0<\varepsilon<r,\ |\xi - x|<r\},$$

$$u^-(x) = \lim_{r\to 0^+} \inf_{u\in S_\varepsilon} \{u(\xi,\eta) : (\xi,\eta)\in \overline{\Omega_\varepsilon},\ 0<\varepsilon<r,\ |\xi - x|<r\},$$

which are bounded functions on $\overline{\Omega}$. The functions u^+ and u^- are a viscosity sub and super solutions to problem (P_o) , respectively.

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Corollary (I.B., A. Briani, H. Ishii)

If (P_o) satisfies the comparison principle, $u^+ = u^- = u_o$ and the convergence of u_ε to u_o is uniform.

What happens if the crucial hypothesis is not satisfied

The crucial assumption on $\gamma^{\pm}$ and $\beta^{\pm}$ for $x \in \overline{\Omega}$

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We consider the very simple problem:

$$\begin{cases} -u_{yy} + u = 1 \quad \text{in } (0,1)\times(0,\varepsilon), \\ u_y(x,\varepsilon) = 1 \ , \ -u_y(x,0) = 0 \ , \ u_x(0,y) = 0, \quad u_x(1,y) = 0 \end{cases}$$

for which the hypotheses on β^+ and β^- are not satisfied.

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for which the hypotheses on β^{+} and β^{-} are not satisfied. The solution is given by

$$u^{\varepsilon}(x,y) = \frac{1}{e^{\varepsilon} - e^{-\varepsilon}} \left(e^{y} + e^{-y} \right) + 1,$$

which is not bounded as ε goes to zero.

Going back to the result of Hale and Raugel i.e. when the "oblique" condition is just the Neumann boundary condition

$$-\Delta u + u = f \text{ in } \Omega_\varepsilon^{HR} \quad \text{and} \quad \frac{\partial u^\varepsilon}{\partial \nu_\varepsilon} = 0 \text{ on } \partial\Omega_\varepsilon^{HR},$$

where $\Omega_\varepsilon^{HR} = \{(x, y) \in \mathbb{R}^N \times \mathbb{R} \,:\, x \in \Omega,\ 0 < y < \varepsilon g(x)\}$, and ν_ε denotes the outward (unit) normal to Ω_ε^{HR}.
Hence with $g^- = 0$, $\gamma_o = 0$ and $k^+(x) = \frac{Dg(x)}{g(x)}$:

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Hence with $g^- = 0$, $\gamma_o = 0$ and $k^+(x) = \frac{Dg(x)}{g(x)}$:
the limit equation will be:

$$\begin{cases} -\mathrm{tr}\begin{pmatrix} D^2 w & 0 \\ 0 & Dg \cdot Dw/g \end{pmatrix} + w = f(x, 0) \text{ in } \Omega, \\ \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases}$$

Even in the case of the Laplacian the result is stronger then Hale and Raugel, because they require g to be C^3

Examples in the Neumann case

We give two very simple examples in the degenerate case with $\Omega_\varepsilon^{HR} = \{(x,y) \in \mathbb{R}^N \times \mathbb{R} \,:\, x \in \Omega,\ 0 < y < \varepsilon g(x)\}$.
Let u_ε be the solution of

$$-\partial^2_{yy}(u_\varepsilon) + u_\varepsilon = f(x,y) \text{ in } \ \Omega_\varepsilon^{HR}, \qquad \partial_{\nu_\varepsilon} u_\varepsilon = 0 \text{ on } \ \partial\Omega_\varepsilon^{HR}.$$

Then, u_ε converges to u_o solution of a first order equation precisely:

$$-\frac{Dg \cdot Du_o}{g} + u_o = f(x,0) \text{ in } \Omega, \qquad \partial_\nu u_o = 0 \text{ on } \ \partial\Omega.$$

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$$-\frac{Dg \cdot Du_o}{g} + u_o = f(x, 0) \text{ in } \Omega, \qquad \partial_\nu u_o = 0 \text{ on } \partial\Omega.$$

Instead, if u_ε is the solution of

$$-\partial^2_{x_1 x_1}(u_\varepsilon) + u_\varepsilon = f(x, y) \text{ in } \Omega_\varepsilon^{HR}, \qquad \partial_{\nu_\varepsilon} u_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon^{HR}$$

it will converge to u_o solution of

$$-(u_o)_{x_1 x_1} + u_o = f(x, 0) \text{ in } \Omega, \qquad \partial_\nu u_o = 0 \text{ on } \partial\Omega.$$

Pucci operator: Uniformly elliptic but fully nonlinear.
Let $0 < \lambda \leq \Lambda$:

$$-\mathcal{M}^+_{\lambda,\Lambda} u_\varepsilon + u_\varepsilon = f(x,y) \text{ in } \Omega_\varepsilon^{HR} \quad \text{and} \quad \frac{\partial u^\varepsilon}{\partial \nu_\varepsilon} = 0 \text{ on } \partial\Omega_\varepsilon^{HR},$$

where

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) := \sup_{\lambda I \leq A \leq \Lambda I} (trA(D^2u)) := \lambda \sum_{e_i \leq 0} e_i + \Lambda \sum_{e_i \geq 0} e_i$$

where e_i are the eigenvalues of D^2u.
In this case the limit equation becomes:

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2w(x)) \quad -\Lambda \left(\frac{Dg(x)\cdot Dw(x)}{g(x)}\right)^+ + \lambda \left(\frac{Dg(x)\cdot Dw(x)}{g(x)}\right)^- +$$

$$+w(x) = f(x,0).$$

Heuristic 2: For the oblique condition $\gamma \cdot \nabla u = \beta$

Intrinsic approach, à la Evans, is to consider the development:

$$u^\varepsilon(x,y) = u^0(x) + \varepsilon u^1(x, y/\varepsilon) + \varepsilon^2 u^2(x, y/\varepsilon) + \cdots ,$$

so that

$$\begin{cases} D^2 u^\varepsilon & = \begin{pmatrix} D_x^2 u^0 & (D_x u_y^1)^{\mathrm{T}} \\ D_x u_y^1 & \varepsilon^{-1} u_{yy}^1 + u_{yy}^2 \end{pmatrix} + o(1), \\ D u^\varepsilon & = (D_x u^0, u_y^1) + o(1), \qquad u^\varepsilon = u^0(x) + o(1). \end{cases}$$

Hence a natural ansatz here, to obtain a PDE for $u^0 = \lim_{\varepsilon \to 0^+} u^\varepsilon$, is to impose that

$$u_{yy}^1(x,y) = 0 \quad \text{for} \quad x \in \overline{\Omega},\ g^-(x) < y < g^+(x).$$

To achieve this, we assume that there is a function $v : \overline{\Omega} \to \mathbb{R}$ such that

$$u^1(x,y) = v(x) y \quad \text{for} \quad y \in \mathbb{R}.$$

Using now the development of the boundary conditions:

$$\begin{cases} \gamma_1^{\pm}(x,y) = \gamma_1^{\pm}(x,0) + k^{\pm}(x)y + o|y|) \\ \beta^{\pm}(x,y) = \beta^{\pm}(x,0) + l^{\pm}(x)y + o(|y|). \end{cases}$$

From the ansatz we obtain, by imposing that the zero order term in the expansion in ε of the boundary condition is zero:

$$0 = \gamma_1^{\pm}(x,0) \cdot Du^0(x) \pm v(x) - \beta^{\pm}(x,0),$$

It yields:

$$v(x) = \beta^{+}(x,0) - \gamma_1^{+} \cdot Du^0(x) = -\beta^{-}(x,0) + \gamma_1^{-} \cdot Du^0(x).$$

This is well defined if condition

$\beta^{+}(x,0) = -\beta^{-}(x,0) := \beta_o(x)$ and $\gamma_1^{+}(x,0) = -\gamma_1^{-}(x,0) := \gamma_o(x).$

holds, and it determines the value for v as

$$v(x) := \beta_o - \gamma_o \cdot Du^0(x),$$

$$u^\varepsilon(x,y) = u^0(x) + \varepsilon u^1(x, y/\varepsilon) + \varepsilon^2 u^2(x, y/\varepsilon) + \cdots ,$$

The value of $u_1(x,y) = (\beta_o - \gamma_o \cdot Du^0(x))y$ has been found just by considering null the zero order term in the expansion in ε of the boundary condition.
Similarly, in order to determine u^2 in terms of u_o..etc..we need to impose that the first order term in the expansion in ε of the boundary condition is zero. And we obtain:

$$u^2(x,y) = \frac{1}{2}(y - g^-(x))^2 w^+(x) + \frac{1}{2}(y - g^+(x))^2 w^-(x)$$

where

$$w^\pm(x) = \frac{g^\pm}{(g^+ - g^-)}(l^\pm(x) - k^\pm(x) \cdot Du^0(x) \mp \gamma_o \cdot Dv),$$

With the knowledge of u^1 and u^2, we are now in a position to guess the limit equation since

$$\begin{cases} D^2 u^\varepsilon & = \begin{pmatrix} D_x^2 u^0 & (D_x u_y^1)^{\mathrm{T}} \\ D_x u_y^1 & \varepsilon^{-1} u_{yy}^1 + u_{yy}^2 \end{pmatrix} + o(1), \\ Du^\varepsilon & = (D_x u^0, u_y^1) + o(1), \qquad u^\varepsilon = u^0(x) + o(1). \end{cases}$$

Of course now that we have the Ansatz

$$u^\varepsilon(x,y) = u^0(x) + \varepsilon u^1(x, y/\varepsilon) + \varepsilon^2 u^2(x, y/\varepsilon) + \cdots,$$

with $u_1(x,y) = (\beta_o - \gamma_o \cdot Du^0(x))y$ and $u^2(x,y) = \frac{1}{2}(y - g^-(x))^2 w^+(x) + \frac{1}{2}(y - g^+(x))^2 w^-(x)$. The Evans test function approach consist, in a very simplified sense, to construct sub and super solutions to the equation at $\varepsilon > 0$, by replacing, in the ansatz, u_o by the test functions of the limit equation. And then proceed.....

The presence of corners with oblique boundary conditions require some attention. Let us give the definition of viscosity solution for problem:

$$F(D^2u, Du, u, (x, y)) = 0 \text{ in } \Omega_\varepsilon$$

satisfying condition

$$\gamma \cdot Du = \beta \text{ on } \quad \partial\Omega_\varepsilon.$$

Intending

$$\gamma^+ \cdot Du^\varepsilon = \beta^+ \text{ on } \partial_T\Omega_\varepsilon \quad \text{and} \quad \gamma^- \cdot Du^\varepsilon = \beta^- \text{ on } \partial_B\Omega_\varepsilon,$$

$$\gamma_L \cdot Du^\varepsilon = \beta_L \text{ on } \partial_L\Omega_\varepsilon.$$

We suppose that
$\gamma^\pm \in C(\overline{\Omega} \times [-1, 1], \mathbb{R}^{N+1})$, $\gamma_L \in C(\partial\Omega \times [-1, 1], \mathbb{R}^{N+1})$ be such that the right structural conditions are satisfied.

Definition of Viscosity Solutions

A bounded function u is a viscosity subsolution if $u^\star$ its upper continuous envelope satisfies: whenever $\phi \in C^2(\overline{\Omega_\varepsilon})$, $\hat{z} = (\hat{x}, \hat{y}) \in \overline{\Omega_\varepsilon}$ and $\max_{\overline{\Omega_\varepsilon}}(u^\star - \phi) = (u^\star - \phi)(\hat{z})$, we must have :

- if $\hat{z} \in \Omega_\varepsilon$,

$$F(D^2\phi(\hat{z}), D\phi(\hat{z}), u^\star(\hat{z}), \hat{z}) \leq 0 \tag{3}$$

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- if $\hat{z} \in \partial\Omega_\varepsilon \setminus ((\partial_{\mathrm{L}}\Omega_\varepsilon \cap \partial_{\mathrm{T}}\Omega_\varepsilon) \cup (\partial_{\mathrm{L}}\Omega_\varepsilon \cap \partial_{\mathrm{B}}\Omega_\varepsilon))$ we have either (3) or
$$\gamma(\hat{z}) \cdot D\phi(\hat{z}) \leq \beta \tag{4}$$

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$$\gamma(\hat{z}) \cdot D\phi(\hat{z}) \leq \beta \tag{4}$$

- if $\hat{z} \in \partial_{\rm L}\Omega_\varepsilon \cap \partial_{\rm B}\Omega_\varepsilon$, we have either (3) or $\gamma_L(\hat{z}) \cdot D\phi(\hat{z}) \leq \beta_L$, or $\gamma^-(\hat{z}) \cdot D\phi(\hat{z}) \leq \beta^-(\hat{z})$,

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$$\gamma(\hat{z}) \cdot D\phi(\hat{z}) \leq \beta \tag{4}$$

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- if $\hat{z} \in \partial_{\mathrm{L}}\Omega_\varepsilon \cap \partial_{\mathrm{T}}\Omega_\varepsilon$., we have either (3), or $\gamma_L(\hat{z}) \cdot D\phi(\hat{z}) \leq \beta_L$ or $\gamma^+(\hat{z}) \cdot D\phi(\hat{z}) \leq \beta^+(\hat{z})$

Definition of Viscosity Solutions

Replacing "max", "$\leq$", and "upper semicontinuous envelope", with "min", "$\geq$", and "lower semicontinuous envelope" respectively, in the above condition yields the right definition of viscosity supersolution. Viscosity solutions are functions which are both viscosity sub and super solutions.

With this definition we do not require the viscosity solution to be continuous.

Because the conditions are on the upper and lower semicontinuous envelope not directly on the solution. The reason being that with corner we don't have comparison principle.

Main existence result (Fixed ε).

Proposition

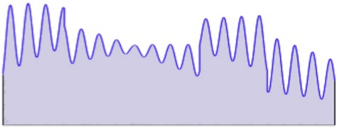
Assume that F is proper. Then, there exist positive constants ε_1 and C_0 such that for each $0 < \varepsilon < \varepsilon_1$, there is a viscosity solution u^ε to

$$\begin{array}{ll} F(D^2u^\varepsilon, Du^\varepsilon, u^\varepsilon, (x,y)) = 0 & \text{in } \Omega_\varepsilon \\ \gamma_1^+ \cdot D_x u^\varepsilon + u_y^\varepsilon = \beta^+ & \text{on } \partial_{\mathrm{T}}\Omega_\varepsilon \\ \gamma_1^- \cdot D_x u^\varepsilon - u_y^\varepsilon = \beta^- & \text{on } \partial_{\mathrm{B}}\Omega_\varepsilon, \\ \gamma \cdot Du^\varepsilon = \beta, & \text{on } \partial_{\mathrm{L}}\Omega_\varepsilon \end{array}$$

and furthermore any solution u^ε will satisfy $\sup_{\overline{\Omega_\varepsilon}} |u^\varepsilon| \leq C_0$.

Bibliography and other projects

- Consider the case where g is not continuous, (disegno)
- $\Omega_\varepsilon = \{(x,y), x \in (0,1), 0 < y < \varepsilon(a(x) + b(x)g(\frac{x}{\varepsilon})\}$, g



periodic.
- Different shapes of thin domains
- More general multi-scale and homogenization problem

We want to recall the works of Arrieta, Pereira, Nogueira, Nakasato, Villanueva-Pesqueira.
But also we would like to understand the relationship with the results on "thin domains" in other contest. e.g. the work of Percivale, Buttazzo, Acerbi (1988) and the sequent ones, or Fonseca, Francfort, G. Leoni..... on "thin elastic films"

Merci de votre attention