Stratified Problems arising from Homogenization of Hamilton-Jacobi equations

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collaborations with

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Outline

Optimal control problems with discontinuities

In the last decade, an important research activity has been focused on Hamilton-Jacobi equations with discontinuities in the data, and in particular on optimal control problems with discontinuities in the cost or the dynamics.

- Important progress have been made when the discontinuities are located on submanifolds of codimension one, see the first part of the book of Barles-Chasseigne and the references therein
- When the discontinuities are located on submanifolds of codimension > 1, the situation is more complex
- *Stratified problems* (Bressan-Hong, Barles-Chasseigne): optimal control problems with discontinuities lying on a union of submanifolds which form a Whitney stratification of $\mathbb{R}^d$, see the second part of the book of Barles-Chasseigne

Goals

Independently, Blanc, Le Bris, Lions and their collaborators have studied homogenization theory in the presence of *local defects* within an otherwise periodic environment, mostly for elliptic equations.

Our purpose: discuss simple examples in which Homogenization or Dimension Reduction with Hamilton-Jacobi equations leads to stratified problems. In particular, we will study periodic homogenization with defects of periodicity.

Several examples

We will tackle three situations in which Homogenization/Dimension Reduction leads to an effective stratified problem

1. with a discontinuity located on a submanifold of codimension one (with S. Oudet and N. Tchou)
2. with a discontinuity located on a submanifold of dimension zero (with C. Le Bris)
3. associated to a stratification of $\mathbb{R}^2$ of the type: $\mathbb{R}^2 = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2$

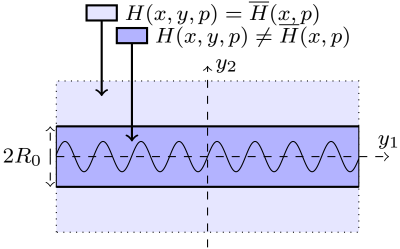
In all situations, a key argument will be the construction of correctors or subcorrectors associated to ergodic problems in unbounded domains.

Outline

The prototypical situation

Goal: study the asymptotic behavior as $\varepsilon \to 0$ of u_ε, the solution of:

$$\alpha\, u_\varepsilon + H\left(x, \frac{x}{\varepsilon}, Du_\varepsilon\right) = 0 \quad \text{in } \mathbb{R}^2. \tag{1}$$



Fixing $(x,p) \in \mathbb{R}^2 \times \mathbb{R}^2$, the function $y \mapsto H(x,y,p)$ is continuous, constant in the light blue region and 1-periodic with respect to y_1 in the darker blue region. The sinusoidal graph is just meant to symbolize the fact that $y \mapsto H(x,y,p)$ is periodic with respect to y_1 in the darker blue region.

Stratification: $\mathcal{M}_1 = \mathbb{R} \times \{0\}, \qquad \mathcal{M}_2 = \mathbb{R}^2 \setminus \mathcal{M}_1$

Assumptions

$$H(x,y,p) = \max_{a\in A}\Big(-p\cdot f(x,y,a) - \ell(x,y,a)\Big)$$

- A is a compact set
- $f : \mathbb{R}^2 \times \mathbb{R}^2 \times A \to \mathbb{R}^2$ and $\ell : \mathbb{R}^2 \times \mathbb{R}^2 \times A \to \mathbb{R}$ are bounded and continuous. For any $(x, y, \tilde{x}, \tilde{y}) \in (\mathbb{R}^2)^4$ and $a \in A$,

  $$|f(x,y,a) - f(\tilde{x},\tilde{y},a)| \le L_f(|x-\tilde{x}| + |y-\tilde{y}|)$$

  Concerning ℓ, there exists a modulus of continuity ω_ℓ s.t. for any $(x, y, \tilde{x}, \tilde{y}) \in (\mathbb{R}^2)^4$ and $a \in A$,

  $$|\ell(x,y,a) - \ell(\tilde{x},\tilde{y},a)| \le \omega_\ell(|x-\tilde{x}| + |y-\tilde{y}|)$$

- **Strong local controllability:** there exists $r > 0$ s.t. for any $x \in \mathbb{R}^2$, $y \in \mathbb{R}^2$, $\{f(x,y,a), a \in A\}$ contains the ball $B_r(0)$
- **Periodicity w.r.t. y_1:** For any $x \in \mathbb{R}^2$ and $a \in A$, $y \mapsto f(x,y,a)$ and $y \mapsto \ell(x,y,a)$ are 1-periodic with respect to y_1
- **Fast variations take place in a strip:** there exist functions $\bar{f} : \mathbb{R}^2 \times A \to \mathbb{R}^2$ and $\bar{\ell} : \mathbb{R}^2 \times A \to \mathbb{R}$ s.t.

  $$f(x,y,a) = \bar{f}(x,a), \quad \text{and} \quad \ell(x,y,a) = \bar{\ell}(x,a) \quad \text{if } |y_2| \ge R_0$$

Homogenization

Set $\mathcal{M}_1 = \mathbb{R} \times \{0\}$ and $\mathcal{M}_2 = \mathbb{R}^2 \setminus \mathcal{M}_1$.

Theorem. (A-Oudet-Tchou)
As $\varepsilon \to 0$, u_ε converges locally uniformly in $\mathbb{R}^2$ to a bounded and Lipschitz function u, the unique solution to the stratified problem:

1. *u is a viscosity solution of*

$$\alpha\, u + \overline{H}(\cdot, Du) = 0 \quad \text{in } \mathcal{M}_2$$

2.
   - *If $\phi \in C^1(\mathbb{R}^2)$ is s.t. $u - \phi$ has a local minimum at $x \in \mathcal{M}_1$, then*

$$\alpha\, u(x) + \max\left(\overline{H}_{1,T}(x, \partial_{x_1}\phi(x)), \overline{H}(x, D\phi(x))\right) \geq 0, \qquad (2)$$

     where $\overline{H}_{1,T} : \mathcal{M}_1 \times \mathbb{R} \to \mathbb{R}$ **is an effective tangential Hamiltonian**

   - *If $\phi \in C^1(\mathcal{M}_1)$ is s.t. $u|_{\mathcal{M}_1} - \phi$ has a local maximum at $x \in \mathcal{M}_1$, then*

$$\alpha\, u(x) + \overline{H}_{1,T}(x, \partial_{x_1}\phi(x)) \leq 0$$

Comments

- Note the difference between the sub and supersolution conditions at $x \in \mathcal{M}_1$
- The theorem holds if $\overline{H}$ is defined independently in $\{x_2 > 0\}$ and $\{x_2 < 0\}$ (discontinuity across $\{x_2 = 0\}$), provided (2) is replaced with

$$\alpha\, u(x) + \max\left(\overline{H}_{1,T}(x, \partial_{x_1}\phi(x)), \overline{H}(x_1, 0_+, D\phi(x)), \overline{H}(x_1, 0_-, D\phi(x))\right) \geq 0$$

An equivalent formulation ($\overline{H}$ possibly discontinuous across $\{x_2 = 0\}$)

Because $\mathcal{M}_1$ is of codimension one, the effective problem is equivalent to:

1. u is a viscosity solution of

$$\alpha\, u + \overline{H}(\cdot, Du) = 0 \quad \text{in } \mathcal{M}_2$$

2.
   - If $\phi \in C^1(\mathbb{R}^2)$ is s.t. $u - \phi$ has a local minimum at $x \in \mathcal{M}_1$, then

$$\alpha\, u(x) + \max\Big(\overline{H}_{1,T}(x, \partial_{x_1}\phi(x)), \overline{H}^{\downarrow}(x_1, 0_+, D\phi(x)), \overline{H}^{\uparrow}(x_1, 0_-, D\phi(x))\Big) \geq 0$$

   - If $\phi \in C^1(\mathbb{R}^2)$ is s.t. $u - \phi$ has a local maximum at $x \in \mathcal{M}_1$, then

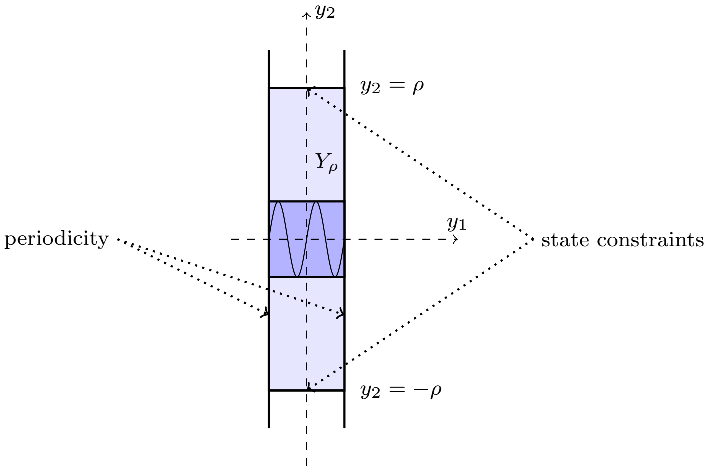
$$\alpha\, u(x) + \max\Big(\overline{H}_{1,T}(x, \partial_{x_1}\phi(x)), \overline{H}^{\downarrow}(x_1, 0_+, D\phi(x)), \overline{H}^{\uparrow}(x_1, 0_-, D\phi(x))\Big) \leq 0$$

where

$$\overline{H}^{\downarrow}(x,p) = \max_{a\in A:\ \overline{f}_2(x,a)\geq 0} \Big(-p\cdot \overline{f}(x,a) - \overline{\ell}(x,a)\Big),$$

$$\overline{H}^{\uparrow}(x,p) = \max_{a\in A:\ \overline{f}_2(x,a)\leq 0} \Big(-p\cdot \overline{f}(x,a) - \overline{\ell}(x,a)\Big)$$

Truncated cell problems for building the effective tangential Hamiltonian $\overline{H}_{1,T}(x, p_1)$, $p_1 \in \mathbb{R}$



The cell $Y = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ and the truncated cell $Y_\rho = (\mathbb{R}/\mathbb{Z}) \times (-\rho, \rho)$

Truncated cell problems for building the effective tangential Hamiltonian

- Given $p_1 \in \mathbb{R}$, the *truncated cell problem* can be seen as an ergodic problem in Y_ρ associated with state constraints on the boundaries $\{y : y_2 = \pm\rho\}$:

$$
\begin{aligned}
H(x, y, p_1 e_1 + D\xi_\rho(x, p_1, y)) &\leq \lambda_\rho(x, p_1) \quad \text{if } y \in Y_\rho, && (3)\\
H(x, y, p_1 e_1 + D\xi_\rho(x, p_1, y)) &\geq \lambda_\rho(x, p_1) \quad \text{if } y \in \overline{Y_\rho} && (4)
\end{aligned}
$$

- It is easy to prove that for a positive K which may depend on p_1 but not on x and ρ, and for all $R_0 < \rho_1 \leq \rho_2$,

$$\lambda_{\rho_1}(x, p_1) \leq \lambda_{\rho_2}(x, p_1) \leq K$$

- For $p_1 \in \mathbb{R}$, the effective tangential Hamiltonian $\overline{H}_{1,T}(x, p_1)$ is defined by

$$\overline{H}_{1,T}(x, p_1) = \lim_{\rho \to \infty} \lambda_\rho(x, p_1)$$

- Important properties of $\overline{H}_{1,T}$ are inherited from the original Hamiltonian, in particular coercivity, convexity and regularity properties.

The related correctors

Theorem. *Let $\xi_\rho(x,p_1,\cdot)$ be a sequence of uniformly Lipschitz continuous solutions of the truncated cell-problem* (3)-(4) *which converges to $\xi(x,p_1,\cdot)$ locally uniformly in Y. Then $\xi(x,p_1,\cdot)$ is a Lipschitz continuous viscosity solution of the following equation posed in Y:*

$$H(x,y,p_1e_1+D\xi(x,p_1,y))=\overline{H}_{1,T}(x,p_1) \tag{5}$$

By subtracting $\xi(x,p_1,0)$ to $\xi_\rho(x,p_1,\cdot)$ and $\xi(x,p_1,\cdot)$, we may assume that

$$\xi(x,p_1,0)=0$$

The following result is reminiscent of [Galise-Imbert-Monneau]:

Theorem. *There exists a sequence $\varepsilon_n\to 0$ s.t. $y\mapsto \varepsilon_n\xi(x,p_1,\frac{y}{\varepsilon_n})$ converges locally uniformly to a Lipschitz function $y\mapsto \Xi(x,p_1,y)$, constant with respect to y_1 and s.t. $\Xi(x,p_1,0)=0$. It is a viscosity solution of*

$$\overline{H}\left(x,\frac{d\Xi}{dy_2}(x,p_1,\cdot)e_2+p_1e_1\right)=\overline{H}_{1,T}(x,p_1),\quad \textit{in } \mathcal{M}_2$$

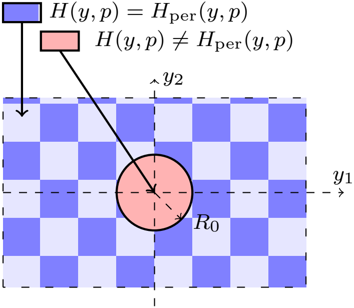
Hence,

$$\overline{H}_{1,T}(x,p_1)\geq \min_{q\in\mathbb{R}}\overline{H}(x,p_1e_1+qe_2)$$

Outline

Local defects within an otherwise periodic environment

$$\alpha\, u_\varepsilon + H\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) = 0 \quad \text{in } \mathbb{R}^d \tag{6}$$



A *local defect* within an otherwise periodic environment.

Remark: Here, we have considered that H depends only on y and p, but the result below holds for $H(x, y, p)$ with a smooth dependence on x.

Assumptions

$$H(y,p) = \max_{a\in A}\Big(-p\cdot f(y,a) - \ell(y,a)\Big)$$

- A is a compact set
- Same boundedness, regularity and controllability assumptions as before
- **Periodicity except near the origin:**

$$f(y,a) = f_{\rm per}(y,a), \quad \text{and} \quad \ell(y,a) = \ell_{\rm per}(y,a) \quad \text{if } |y| \geq R_0$$

Set

$$H_{\rm per}(y,p) = \max_{a\in A}\Big(-p\cdot f_{\rm per}(y,a) - \ell_{\rm per}(y,a)\Big)$$

Homogenization

Set $\mathcal{M}_0 = \{0\}$ and $\mathcal{M}_d = \mathbb{R}^d \setminus \mathcal{M}_0$.

Theorem. (A-Le Bris) *As $\varepsilon \to 0$, u_ε converges locally uniformly in $\mathbb{R}^d$ to a bounded and Lipschitz function u, the unique solution to the stratified problem:*

1. *u is a viscosity solution of*

$$\alpha\, u + \overline{H}(Du) = 0 \quad \text{in } \mathcal{M}_d$$

   with the effective Hamiltonian $\overline{H}$ obtained in periodic homogenization, *see Lions, Papanicolaou, Varadhan*

2. - *The condition*

$$\alpha\, u(0) + E \leq 0 \tag{7}$$

     holds, **where E is an effective Dirichlet datum**
   - *If $\phi \in C^1(\mathbb{R}^d)$ is s.t. $u - \phi$ has a local minimum at the origin, then*

$$\alpha\, u(0) + \max\left(E, \overline{H}(D\phi(0))\right) \geq 0 \tag{8}$$

Remark: Since $\overline{H}$ is convex and u is Lipschitz continuous, we also know that if $\phi \in C^1(\mathbb{R}^d)$ is s.t. $u - \phi$ has a local maximum at the origin, then

$$\alpha\, u(0) + \overline{H}(D\phi(0)) \leq 0 \tag{9}$$

Comments 1/2

1. The effective problem may be seen as a weak formulation (*stratified formulation*) of a Dirichlet boundary value problem comprising an Hamilton-Jacobi equation in the singular open set $\mathcal{M}_d = \mathbb{R}^d \setminus \{0\}$ and the effective Dirichlet boundary condition $u(0) = -E/\alpha$.

2. This formulation is stronger than the one due to H. Ishii, which expresses that at the origin, u is both a viscosity subsolution of

$$\alpha\, u(0) + \min(\overline{H}(Du(0)), E) \leq 0$$

   and a viscosity supersolution of

$$\alpha\, u(0) + \max(\overline{H}(Du(0)), E) \geq 0.$$

3. The supersolution condition (8) coincides with that of Ishii, whereas the subsolution condition (7) together with (9) is stronger

4. It is well known that there is no uniqueness for Ishii's formulation of a Dirichlet condition at $\{0\}$, whereas the stratified formulation enjoys a comparison principle and uniqueness.

Comments 2/2

1. We will soon see that
$$E \geq \min_{p\in\mathbb{R}^d} \overline{H}(p)$$

2. If $E = \min_{p\in\mathbb{R}^d} \overline{H}(p)$, then the effective constant does not show up in the effective problem, i.e. the Dirichlet condition is irrelevant

3. The homogenization of periodic Hamilton-Jacobi equations in the presence of a defect has been first addressed by P-L. Lions and P. Souganidis, (lectures by P-L. Lions at Collège de France), but the assumptions made therein implied that
$$E = \min_{p\in\mathbb{R}^d} \overline{H}(p),$$
so their effective problem did not involve any boundary condition

4. Thanks to their strong assumptions, Lions and Souganidis were able to get accurate information on correctors, which we will not find with our more general assumptions

Truncated cell problems for building the effective Dirichlet datum

- Given a parameter $\rho > R_0$ which will eventually tend to $+\infty$, the truncated cell is $B_\rho(0)$
- There exists a unique real nomber E^ρ s.t. there exists w^ρ satisfying

$$\begin{array}{rcll} H(0,y,Dw^\rho) & \leq & E^\rho & \text{in } B_\rho(0) \\ H(0,y,Dw^\rho) & \geq & E^\rho & \text{in } \overline{B_\rho(0)} \end{array}$$

  in the sense of viscosity (ergodic problem with state constraints). We can always suppose that $w^\rho(0) = 0$.

- $$\rho_1 \geq \rho_2 \quad \Rightarrow \quad -\min_{(y,a)\in\mathbb{R}^d\times A} \ell(y,a) \geq E^{\rho_1} \geq E^{\rho_2}$$

  Hence, we can define
  $$E = \lim_{\rho\to\infty} E^\rho.$$

The corrector associated to the defect

- Since $w^\rho(0)=0$ and w^ρ is Lipschitz continuous on $\overline{B_\rho(0)}$ with a Lipschitz constant independent of ρ, we may construct by Ascoli-Arzelà theorem and a diagonal extraction argument a sequence $\rho_n \to +\infty$, s.t. w^{ρ_n} tends to some function w locally uniformly in $\mathbb{R}^d$. We then see that $w(0)=0$ and w is a Lipschitz continuous viscosity solution of

$$H(y, Dw) = E \quad \text{in } \mathbb{R}^d.$$

- Consider $w_\varepsilon : x \mapsto \varepsilon w(\frac{x}{\varepsilon})$. It is clearly a viscosity solution of $H(\frac{x}{\varepsilon}, D_x w_\varepsilon) = E$, and it is Lipschitz continuous with the same constant as w. Hence, after the extraction of a sequence, w_ε converges locally uniformly to some Lipschitz function W on $\mathbb{R}^d$, which is a viscosity solution of

$$\overline{H}(DW) = E, \quad \text{in } \mathcal{M}_d = \mathbb{R}^d \setminus \{0\}$$

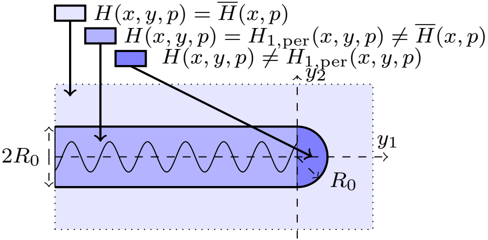
This implies

$$E \geq \min_{p\in\mathbb{R}^d} \overline{H}(p).$$

Outline

A typical situation

$$\alpha\, u_\varepsilon + H\left(x, \frac{x}{\varepsilon}, Du_\varepsilon\right) = 0 \quad \text{in } \mathbb{R}^2. \tag{10}$$



Fixing $(x,p) \in \mathbb{R}^2 \times \mathbb{R}^2$, the function $y \mapsto H(x,y,p)$ is continuous, constant in the light blue region and 1-periodic w.r.t. y_1 in the intermediate blue region. The sinusoidal graph is just meant to symbolize the fact that $y \mapsto H(x,y,p)$ is periodic with respect to y_1 in the intermediate blue region. Note also that $H_{1,\mathrm{per}}(x,y,p) = \overline{H}(x,p)$ if $|y_2| \geq R_0$.

Set $\mathcal{M}_0 = \{0\}$, $\mathcal{M}_1 = (-\infty, 0) \times \{0\}$ and $\mathcal{M}_2 = \mathbb{R}^2 \setminus (\mathcal{M}_1 \cup \mathcal{M}_0)$.

Assumptions

$$H(y,p) = \max_{a\in A}\Big(-p\cdot f(y,a) - \ell(y,a)\Big)$$

- A is a compact set
- Same boundedness, regularity and controllability assumptions as before
- **The fast variations are localized:** Set

$$\Omega = \Big((-\infty,0]\times(-R_0,R_0)\Big)\bigcup\Big\{|y|\le R_0\Big\}.$$

  There exist functions $\bar f:\mathbb{R}^2\times A\to\mathbb{R}^2$ and $\bar\ell:\mathbb{R}^2\times A\to\mathbb{R}$ s.t.

$$f(x,y,a)=\bar f(x,a),\quad \text{and}\quad \ell(x,y,a)=\bar\ell(x,a)\quad \text{if } y\notin\Omega$$

- **Periodicity w.r.t. y_1:** There exist functions $f_{1,\rm per}:\mathbb{R}^2\times\mathbb{R}^2\times A\to\mathbb{R}^2$ and $\ell_{1,\rm per}:\mathbb{R}^2\times\mathbb{R}^2\times A\to\mathbb{R}$, 1-periodic in y_1, s.t. $f(x,y,a)$ and $\ell(x,y,a)$ coincide with $f_{1,\rm per}(x,y,a)$ and $\ell_{1,\rm per}(x,y,a)$ if $y_1<0$

Homogenization

Set $\mathcal{M}_0 = \{0\}$, $\mathcal{M}_1 = (-\infty, 0) \times \{0\}$ and $\mathcal{M}_2 = \mathbb{R}^2 \setminus (\mathcal{M}_1 \cup \mathcal{M}_0)$.

Theorem. *u_ε converges locally uniformly in $\mathbb{R}^2$ to a bounded and Lipschitz function u, which is the unique solution to the following stratified problem associated to $\mathbb{R}^2 = \mathcal{M}_2 \cup \mathcal{M}_1 \cup \mathcal{M}_0$:*

1. *u is a viscosity solution of*

$$\alpha\, u + \overline{H}(\cdot, Du) = 0 \quad \text{in } \mathcal{M}_2. \tag{11}$$

2.
   - *If $\phi \in C^1(\mathbb{R}^2)$ is s.t. $u - \phi$ has a local minimum at $x \in \mathcal{M}_1$, then*

$$\alpha\, u(x) + \max\left(\overline{H}_{1,T}(x, \partial_{x_1}\phi(x)), \overline{H}(x, D\phi(x))\right) \geq 0. \tag{12}$$

   where $\overline{H}_{1,T} : \overline{\mathcal{M}_1} \times \mathbb{R} \to \mathbb{R}$ is the effective tangential Hamiltonian

   - *If $\phi \in C^1(\mathcal{M}_1)$ is s.t. $u - \phi$ has a local maximum at $x \in \mathcal{M}_1$, then*

$$\alpha\, u(x) + \overline{H}_{1,T}(x, \partial_{x_1}\phi(x)) \leq 0. \tag{13}$$

3.
   - *If $\phi \in C^1(\mathbb{R}^2)$ is s.t. $u - \phi$ has a local minimum at 0, then*

$$\alpha\, u(0) + \max\left(E, \overline{H}_{1,T}(0, \partial_{x_1}\phi(0)), \overline{H}(0, D\phi(0))\right) \geq 0, \tag{14}$$

   where E is the effective Dirichlet datum

   -

$$\alpha\, u(0) + E \leq 0. \tag{15}$$

Ingredients in the effective problems

A kind of recursivity. The third situation combines the features of the first two examples:

- periodicity along a half-line instead of a full line
- the periodicity is broken near the origin
- The arguments will be organized in a recursive manner

Effective Hamiltonians and correctors

- For $x \in \mathcal{M}_1$, the effective tangential Hamiltonian $\overline{H}_{1,T}(x, p_1)$ is the same as the one constructed in the first example. The related correctors $y \mapsto \xi(x, p_1, y)$ are the same as well
- The effective Dirichlet datum E at 0 is constructed in the same way as in the second example. The related corrector w is obtained in the same way as well
- In addition to the inequalities

$$E \geq \min_{p \in \mathbb{R}^2} \overline{H}(0, p) \quad \text{and} \quad \overline{H}_{1,T}(x, p_1) \geq \min_{q \in \mathbb{R}} \overline{H}(x, p_1 e_1 + q e_2),$$

we get

$$E \geq \min_{p_1 \in \mathbb{R}} \overline{H}_{1,T}(0, p_1)$$

Sketch of the proof

$$\overline{u}(x) = \limsup_{\varepsilon\to 0} u_\varepsilon(x),$$
$$\underline{u}(x) = \liminf_{\varepsilon\to 0} u_\varepsilon(x)$$

1. The first step is devoted to obtaining the viscosity inequalities satisfied by $\underline{u}$ and $\overline{u}$ at $x \in \mathcal{M}_1$. The techniques are the same as in [A-Oudet-Tchou]
2. To prove that $\overline{u}$ satisfies $\alpha\,\overline{u}(0) + E \le 0$, we rely on Evans' method of perturbed test-functions using either w or the approximate corrector w^ρ
3. A delicate part is to prove that if $\phi \in C^1(\mathbb{R}^2)$ is s.t. $\underline{u} - \phi$ has a local minimum at 0, then

   $$\alpha\,\underline{u}(0) + \max\left(E, \overline{H}_{1,T}(0, \partial_{x_1}\phi(0)), \overline{H}(0, D\phi(0))\right) \ge 0$$

   The main idea is to construct subcorrectors by combining in a suitable way the correctors w and $\xi(0, p_1, \cdot)$ associated to the Hamiltonian $H_{1,\mathrm{per}}(0,\cdot)$
4. Convergence is obtained by using the comparison result for stratified solutions

Subcorrectors at $x = 0$

Why just subcorrectors?
Because we only want to prove a supersolution property at 0 by means of Evans' method.

Lemma *Consider* $p \in \mathbb{R}^2$. *If* $\max(E, \overline{H}_{1,T}(0, p_1)) < \overline{H}(0, p)$, *then there exists a Lipschitz function* $\chi : \mathbb{R}^2 \to \mathbb{R}$ *such that*

$$\chi(y) \le p \cdot y, \quad \text{for all } y \in \mathbb{R}^2, \tag{16}$$

$$H(0, y, D\chi(y)) \le \overline{H}(0, p), \quad \text{in } \mathbb{R}^2, \tag{17}$$

where (17) *is understood in the viscosity sense.*

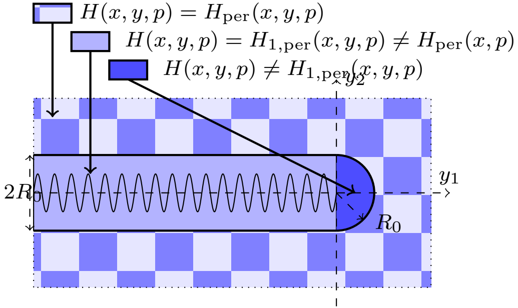
Lemma *Consider* $p \in \mathbb{R}^2$.
If $\max(E, \overline{H}(0, p)) < \overline{H}_{1,T}(0, p_1)$ *or* $E < \overline{H}(0, p) = \overline{H}_{1,T}(0, p_1)$, *then there exists a Lipschitz function* $\chi : \mathbb{R}^2 \to \mathbb{R}$ *such that*

$$\chi(y) \le p \cdot y, \quad \text{for all } y \in \mathbb{R}^2, \tag{18}$$

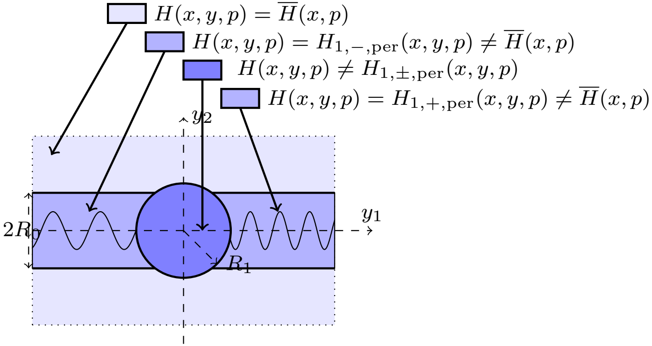
$$H(0, y, D\chi(y)) \le \overline{H}_{1,T}(0, p_1), \quad \text{in } \mathbb{R}^2, \tag{19}$$

where (19) *is understood in the viscosity sense.*

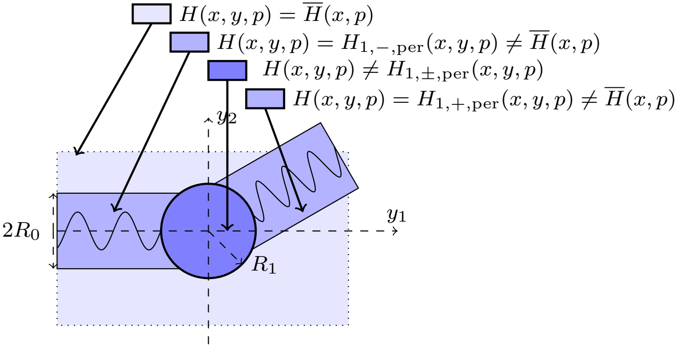
Other cases can be addressed with the same techniques 1/2



Other cases can be addressed with the same techniques 2/2



Cases when I was not able to construct subcorrectors



I was not able to construct subcorrectors, because there is an issue of compatibility between $\xi_{1,+}$ and $\xi_{1,-}$.

A few references

- G. Barles and E. Chasseigne, *On modern approaches of Hamilton-Jacobi equations and control problems with discontinuities—a guide to theory, applications, and some open problems*,, Birkhäuser/Springer, Cham, [2024]
- A. Bressan and Y. Hong, *Optimal control problems on stratified domains*, Netw. Heterog. Media, 2 (2007), pp. 313–331.
- G. Galise, C. Imbert, and R. Monneau, *A junction condition by specified homogenization and application to traffic lights*, Anal. PDE, 8 (2015), pp. 1891–1929.
- P.-L. Lions, *Équations elliptiques ou paraboliques, et homogénéisation précisée, Lectures at Collège de France 2013/14.*
- X. Blanc, C. Le Bris, and P.-L. Lions, *On correctors for linear elliptic homogenization in the presence of local defects*, Comm. Partial Differential Equations, 43 (2018), pp. 965–997.

A few references

Y. Achdou and C. Le Bris, *Homogenization of some periodic Hamilton-Jacobi equations with defects*, Comm. Partial Differential Equations, 48 (2023), pp. 944–986.

Y. Achdou, S. Oudet, and N. Tchou, *Effective transmission conditions for Hamilton-Jacobi equations defined on two domains separated by an oscillatory interface*, J. Math. Pures Appl. (9), 106 (2016), pp. 1091–1121.